Delivering Excellent Learning and Teaching Through

Mental Maths Strategies

Dumfries & Galloway COUNCIL

Contents

Director's Foreword

Dear colleague,

The teaching of mental maths has been identified as a development priority at a national level and we, in Dumfries and Galloway, find ourselves faced with the same challenges as those in the rest of the country. If we are to meet these challenges and raise attainment for all our learners we must do so through sound pedagogy and a structured, progressive development which is understood and applied consistently in all our schools.

To support practitioners in this goal, the authority's Numeracy Working Group, has developed this document and will, in the coming session, support its full implementation in schools with focused professional learning on the practical use of the identified mental maths strategies.

The document gives valuable exemplification of these strategies and activities which will support the learning and teaching of them. However, it is not a programme of study and is flexible enough to allow schools to use their own resources to ensure that learners are given a broad range of experiences in their learning.

The stated aim, from the rationale, is to have the Dumfries and Galloway Mental Maths Strategies document fully embedded in all our schools by August 2017 at the latest with the majority of learners achieving at or above the expected attainment levels contained within it.

Given the commitment and ability of the dedicated professionals within all our schools, I am confident that this will be the case.

I would also like to thank the members of the Numeracy Working Group for their efforts in producing this valuable support to schools.

Colin Grant,
Director, Education Services

Rationale:

Across the authority schools have identified the teaching of mental maths as a key challenge and this has also been recognised in recent standardised assessment results. The development of mental maths strategies, therefore, has been included by many schools in their 2012 – 2013 School Improvement Plans.

This has been reflected at a national level with the successes and challenges which were highlighted in the SSLN Report 2012.

To support schools, the Numeracy Working Group gave priority in their work to developing guidance and a line of progression (from Early to Third Level) in mental maths strategies. This document is the outcome of this development.

From the outset, it was decided that high but achievable goals should be set with a view to raising attainment through the use of the strategies (e.g. Partitioning using near doubles). These must be practised from Early Level onwards to enable learners to build on prior knowledge.

It is the development of these strategies, which will support learners in becoming skilled in deciding upon which one is the most efficient to use given the calculation to be solved. This is of vital importance to their development.

It is important however that practitioners view the levels as guidance and move the learners through them at an appropriate pace and with an appropriate level of challenge (e.g. some P7 pupils may very well be working from the 3rd Level line of progression).

Each line of progression should be viewed as developing over a given time scale before it can be fully effective in raising attainment. At a recent National Numeracy Network meeting it was stated that the Scottish Government has an aim to eradicate innumeracy by the year 2017 and it would seem to be reasonable then that full implementation of the Dumfries and Galloway Mental Maths Strategy should mirror this expectation.

This document will go some way to addressing this challenge through raising attainment and developing essential life skills (BtC 4) for ***all*** learners.

However, the activities and exemplification of progression through them is to support practitioners in their own thinking and should not be viewed as a programme of study.

Mental strategies bridge many of the Maths and Numeracy Es and Os and this document deliberately does not overtly link the strategies to specific ones. However, they strongly link to the identified Es and Os listed below.

Estimating and Rounding –
MNU 0-01a, MNU 1-01a, MNU 2-01a, MNU 3-01a

Number and Number Processes –
MNU 0-02a, MNU 1-02a, MNU 2-02a
MNU 0-03a, MNU 1-03a, MNU 2-03a, MNU 3-03a
MNU 2-03b, MNU 3-03b
MNU MNU 2-04a, MNU 3-04a

Fractions, Decimal Fractions and Percentages -
MNU 0-07a, MNU 1-07a, MNU 2-07a, MNU 3-07a
MNU 1-07b, MNU 2-07b
MNU 3-08

Introduction

The ability to calculate in your head is an important part of mathematics. It is also an essential part of coping with society's demands and managing everyday events.

This Mental Strategies document has been adapted from *a National Strategies Primary (Department of Education England and Wales)* publication entitled ***'Teaching Learners To Calculate Mentally (2010)'***

The overall aim of the Dumfries and Galloway Mental Strategies document is to assist planning, learning and teaching by:

- listing the number facts that learners are expected to recall rapidly
- setting out expectations for the types of calculations that learners should be able to do mentally
- identifying the mental methods that might be taught to learners to help them to calculate accurately and efficiently
- suggesting a range of suitable classroom activities and resources to help learners to understand and practise calculation methods.

The document will be in two parts covering:

Part 1:

Principles of Teaching Mental Calculation:
This promotes a broad interpretation of mental calculation and identifies principles that underpin teaching: for example, encouraging learners to share their mental methods, to choose efficient strategies and to use informal jottings to keep track of the information they need when calculating. It also looks at the role of tests and questioning.

Line of Progression:
This describes the progression in the number facts that learners should derive and recall, the calculations that they are expected to do mentally and the range of calculation strategies or methods which they can draw on. These are shaded to indicate progression within the level.

Part 2:

Mental Strategies and Activities:
This sets out the main strategies for adding, subtracting, multiplying and dividing mentally. It describes activities to support teaching of these strategies and typical problems.


1
2
3
4

Principles of Teaching Mental Calculation

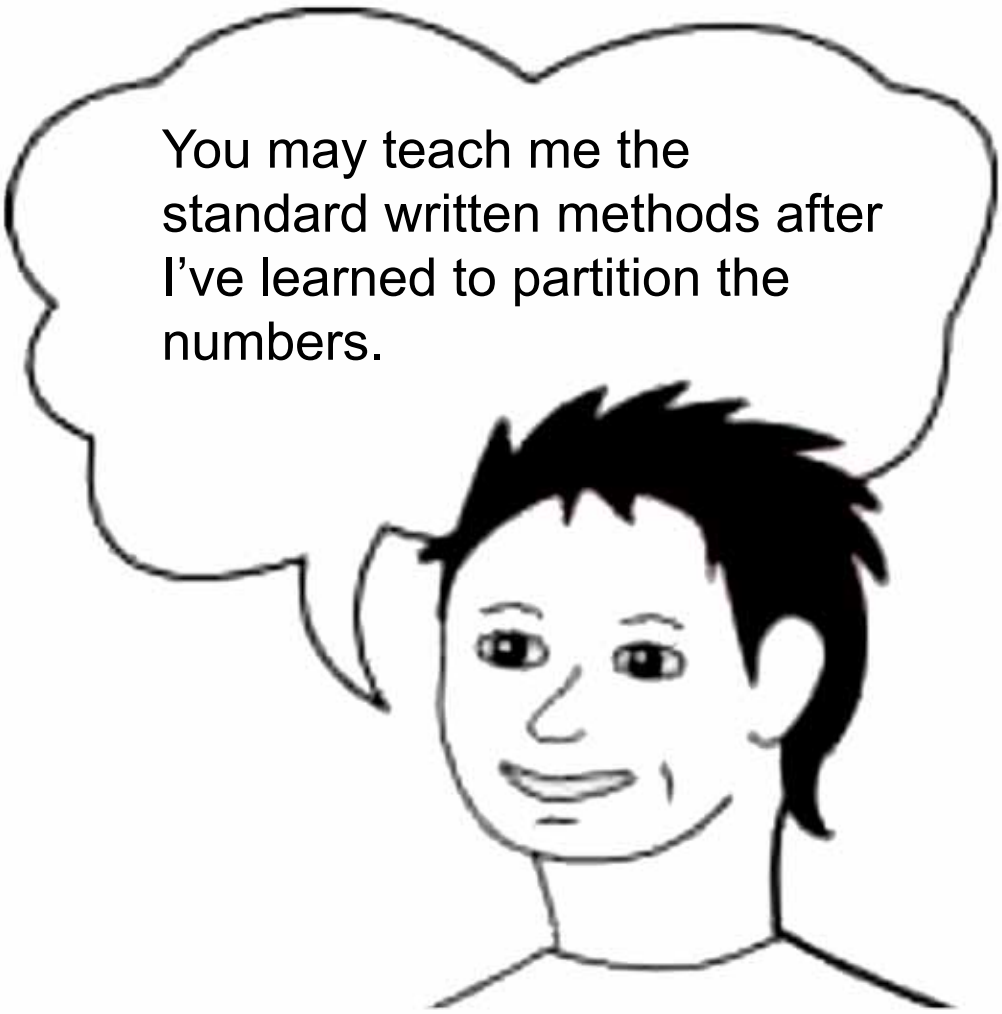
Standard Written Method or Mental Calculation?

Written recording can be seen as a thinking tool, a communication tool, and a reflective tool. The informal jottings of students are to be encouraged as a way to capture their mental processes so that their ideas can be shared with others.

Learners should not be exposed to standard written methods until they have had appropriate experience of using mental strategies. Premature exposure to working forms restricts learners' ability and desire to use mental strategies.

This inhibits their development of number sense. However, in time, written methods must become part of a learner's calculation repertoire.

New Zealand Number Framework



Is mental calculation the same as mental arithmetic?

For many people, mental calculation is about doing arithmetic; it involves rapid recall of number facts (eg knowing your number bonds to 20 and the multiplication tables to 10 x10).

Rapid recall of number facts is one aspect of mental calculation but there are others. This involves presenting learners with calculations in which they have to work out the answer using known facts and not just recall it from a bank of number facts that are committed to memory. Learners should understand and be able to use the relationship between the four operations and be able to construct equivalent calculations that help them to carry out such calculations.

Learning key facts by heart enables learners to concentrate on the calculation which helps them to develop calculation strategies. Using and applying strategies to work out answers helps learners to acquire and so remember more facts. Many learners who are not able to recall key facts often treat each calculation as a new one and have to return to first principles to work out the answer again. Once they have a secure knowledge of some key facts, and by selecting problems carefully, learners can be supported in understanding that, from the answer to one problem, other answers can be generated.

Teaching mental calculation leads to the following teaching principles:

- every day is a mental mathematics day with regular time committed to teaching mental calculation strategies.
- provide practice time with frequent opportunities for learners to use one or more facts that they already know to work out more facts.
- introduce practical approaches and jottings, with models and images, which learners can use to carry out calculations as they secure mental strategies.
- engage learners in discussion when they explain their methods and strategies to others.

Revisiting mental work at different times in the daily mathematics lesson, or even devoting a whole lesson to it from time to time, helps learners to generate confidence in themselves and a feeling that they control calculations rather than calculations controlling them. Find opportunities to introduce short periods of mental calculation in other lessons or outside lesson when queuing for P.E., the dining hall etc.

Regular short practice keeps the mind fresh. Mental calculation is one of those aspects of learning where, *if you don't use it you will end up losing it!*

What's special about mental calculation?

A feature of mental calculation is that a type of calculation can often be worked out in several different ways. Which method is best will depend on the numbers involved, the age of the learners and the range of methods that they are confident with. This feature differs from a standard written method: the strength of a standard method is that it is generally applicable to all cases irrespective of the numbers involved. For example, the following calculations would all be treated in the same way if the decomposition method of subtraction were used:

61 – 4, 61 – 41, 61 – 32, 61 – 58, 61 – 43

If carried out mentally, each calculation would be done in a different way.

There would be an expectation that, by the end of P5, most learners would be able to do these kinds of calculations mentally, applying strategies to reflect their confidence with an understanding of the alternative approaches. The clear advantages are that develop a much stronger 'number sense', better understanding of place value and more confidence with numbers and the number system.

The underlying teaching principle here is to:

- ensure that learners can confidently add and subtract any pair of two digit numbers mentally, using jottings to help them where necessary.

How do I help learners develop a range of mental strategies?

Learners are likely to be at different stages in terms of number facts that they have committed to memory and the strategies available to them for working out other facts. This document is intended to help staff be more aware of the range of possible strategies to be taught and used as learners progress in mathematics.

This should allow staff to be:

- in a better position to recognise the strategies learners are using when they calculate mentally.
- able to draw attention to and model a variety of the strategies used by the learners.
- able to make suggestions about 'next steps' to learners that will move them onto more efficient strategies.

To enable learners to learn and draw on a range of mental methods, we need to raise their awareness and understanding of the range of possible strategies, develop their confidence and fluency by practising using the strategies, and help them to choose from the range the most efficient method for a given calculation.

The principle here is to teach a mental strategy explicitly but in addition invite learners to suggest an approach and to explain their methods of solution to others. This has the advantages that:

- learners become familiar with an approach they can call their own.
- learners, through explaining to others, clarify their own thinking.
- learners who are listening develop their awareness of the range of possible methods.
- the activity can lead to a discussion of which methods are the most efficient.

Can mental calculations involve practical equipment?

Hands on learning is important
Mental calculations involve visualising, imagining and working things out in your head. However, learners will not be able to visualise and 'see' how something works if they have not had any practical experiences to draw upon or been shown any models and images that support the approaches taught. Resources for this can include – counters, interlocking cubes, coins, counting sticks, bead strings, Rekenrek, number lines, Numicon, interactive whiteboard, 100-squares, place-value cards, structural apparatus like base 10 blocks, diagrams of shapes divided into fractional parts, and so on.

The underlying teaching principle here is to:

- provide suitable equipment for learners to manipulate and explore how and why a calculation strategy works, and that helps them to describe, visualise and 'see' the method working.

Selecting when and how to use and withdraw resources and visual images is a key part of teaching.
This involves planning how best to construct a blend of teaching approaches that are selected and designed to match intended learning intentions, success criteria and learners' needs.
It is asking:

"What is it I want to achieve?"
"Is it fit for purpose?"
"Is there a better way?"

Visualising is important in mathematics.
The ability to visualise representations, pictures or images and then adapt or change them is an important tool (eg when problem solving, pattern spotting and reasoning in maths). Like everything else it is a skill that can be acquired through practice and this involves the use of practical equipment and tactile resources. This can take place at all ages and Curriculum for Excellence (CfE) levels. Young learners recognise pictures and representations of objects or people; they learn to describe something they have seen but cannot see at that time; they associate an image with an object, stimulus or emotion. Mathematics needs to focus on developing these skills. Practising visualisation helps to develop the brain's capacity to 'see' and to 'draw pictures' in the same way that we need to practice this on paper.

Can mental calculations involve pencil and paper?

Young learners often support their early mathematical thinking through the use of 'mark making' involving drawings, writing, tally-marks and invented or standard symbols such as numerals. 'Informal' recording helps represent the ideas and steps undertaken in coming to an answer. Learners should not just record the learning in a neat and complete way after it's been worked out. 'Informal', on-going recording is an extension of thinking. Pencil and paper can support mental calculation in a variety of ways:

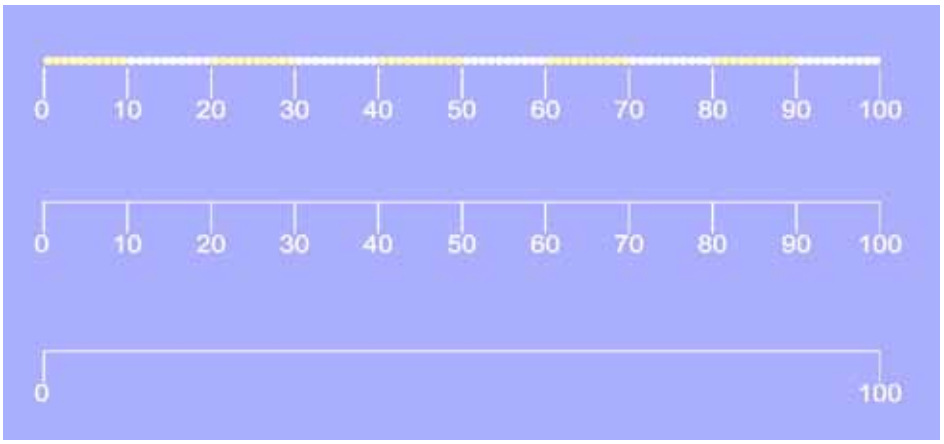
- through jottings – informal notes during steps in a calculation
- through recording for an explanation of the method used. A written account can help learners begin to use appropriate notation and form the basis for developing more formal written methods.
- through models and diagrams that support the development of mental imagery (eg number lines)

The underlying teaching principle here is to:

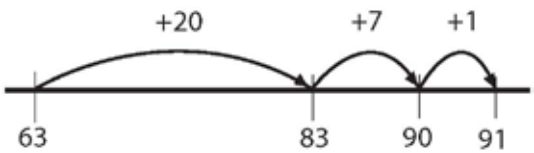
- encourage learners to make jottings as they work and to recognise how these can support their thinking; model this process for them and distinguish between a presentation and a jotting.

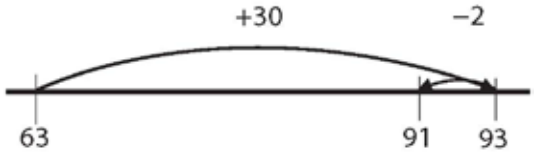
Using an empty number line.

The empty number line is a powerful model for developing learners' calculation strategies and developing their understanding. A progression from practical bead strings to number lines with landmark numbers (eg those with intervals representing multiples of 5 or 10 marked) can help learners develop the image of the number line in their head.



Here are two examples of the calculation 63 + 28 = 91 represented on the empty number line:





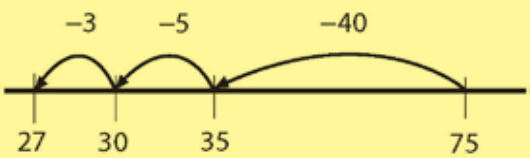
Both of these examples model common mental approaches using counting on. Each starts at the larger number (63).

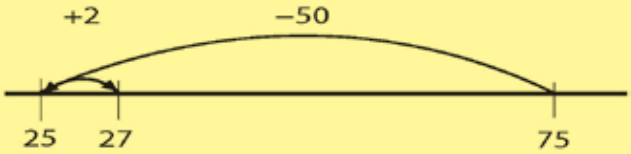
In the first example, 28 is partitioned into 20 + 8. The tens are added first, then the units are added by *'bridging through a multiple of 10'* (in this case 90).

In the second example, a first step is counting on more than 28, to the next higher multiple of 10 (in this case 30). By adjusting (in this case by 2) the correct answer is achieved. This method is often called the *'compensation method'* .

In either case, the answer of 91 appears as a position on the number line.

One model of subtraction, the inverse of addition, involves counting back shown in the following two examples counting back from 75.

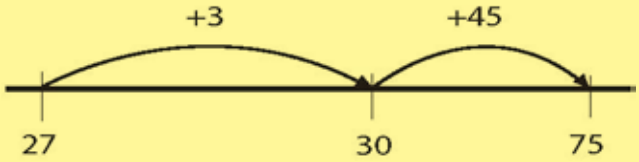




In the first example 48 Is partitioned into 40 + 8. The tens are subtracted first, then the units *'bridging through a multiple of 10'* (In this case 30).

In the second example a first step is counting back too many (the next higher multiple of ten which 50). By adjusting (in this case by 2) the correct answer is achieved. In either case the answer of 27 appears as a position on the number line.

Subtraction is also the difference between two numbers. The difference can be represented as the distance between the two numbers on the number line. This distance is usually worked out by counting on from the smaller to the larger number, *'bridging through a multiple of ten'.* In this case the answer to 75 – 27 is equivalent to 45 + 3 = 48 represented by the two jumps and not a point on the number line. This counting on method is often referred to as *'shopkeeper's method'* because it is like a shop assistant counting out change.



Many learners find it easier to subtract by counting on rather than by counting back.

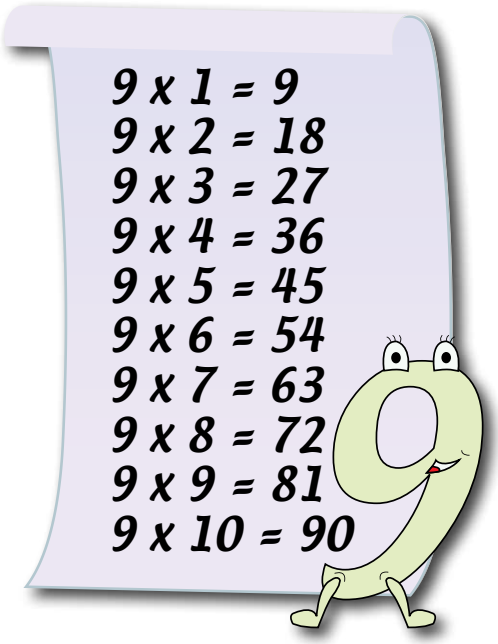
How do pupils respond efficiently to questioning?

Once a mental strategy has been introduced to learners, there comes a time to encourage them to speed up their responses and either use more efficient strategies or expand their range.
The traditional mental maths assessment involves a set of unseen questions. One worthwhile alternative is to give learners examples of the type of questions 10 minutes in advance, so that they can think about the most efficient way to answer the questions. The purpose of this preparation time is not to try to commit answers to memory but to sort the questions into those they 'know' the answer to, and those that they need to work out. Pairs of learners can talk about their 'working out' methods and after the assessment the whole class can spend some time discussing the strategies they used.
Collecting the questions, then giving learners the assessment with the questions in a random order, also encourages attention to strategies. The same assessment can be used at a different time for learners to try to better their previous score. 'Route 136' may be one example (rapid recall of tables and number bonds) that could be used in class.

Another example could be to give learners a multiplication fact they all know. They each record this on their paper so they can refer to it throughout the assessment.

For example: 6 x 3 =18

The 10 questions would all relate to the recorded number fact. The class agrees a set time that will be allowed for each question. The questions can vary in difficulty over the assessment so that all learners can fully engage with it.



For example the questions might be:

6 x 30 =

12 x 3 =

6 x 1.5 =

12 x 6 =

12 x 0.3 =

The questions can also involve inverse operations or could follow a particular set of facts that needs consolidation.

After the assessment the different methods and strategies can be compared and discussed in groups, with the learners identifying those questions that they found most difficult to answer. At the end of the lesson a target time per question is set for the next challenge and particular question types are identified for further consolidation in preparation for the next assessment.

The underlying teaching principle here is:

- encourage learners to compete against themselves, aiming to better their previous performance.

When a mental assessment is marked, there is a chance to discuss any errors made and why they happened. Did the learners guess the answer rather than work it out or did they make a careless error in trying to answer within the time limit? Did the error indicate a genuine misunderstanding which needs to be followed up as a 'next step'? Discussion of the methods used can help to reveal why the error was made. Learners should be supported in understanding that most of us make mistakes at first when learning a new skill but that thinking about the mistake and how and why it was made helps us to improve that skill (Dumfries and Galloway Numeracy Strategy 3-18 – Emotional Environment).

The underlying teaching principle here is:

- encourage the learners to discuss their mistakes and challenges in a positive way so that they learn from them and share the ownership of targets to help manage and recognise their rate of progress.

How do I make mental calculation practice fun?

Practice is essential if learners are to become fluent and confident with mental strategies. The planned and considered use of games and puzzles can support learners in developing their mental strategies and also, importantly, sustain their interest and motivation to continually improve their learning.

Some of these games involve recall of facts while others involve more complex calculations. The best of these also involve an element of reasoning. Many of these can be found on Glow both:
locally (http://glo.li/WCvDyy)
nationally (http://glo.li/WCw0Jz or http://glo.li/k5HrMh).

The following are only two examples of these hyperlinked sites.

http://www.ictgames.com/resources.html http://glo.li/XJjkyZ

The underlying teaching principle here is:

- make mental calculation practice interesting and enjoyable.

How should Learners respond to oral questions?

The method of asking a question and inviting a volley of hands to go up has several drawbacks for learners who are working out answers. It emphasises the rapid, the known, over the derived – learners who 'know' are the first to answer, while those who are collecting their thoughts are distracted by others straining to raise hands.

Consistent AifL approaches, which may include the following should be in evidence in all classrooms:

- insisting that no-one puts a hand up until the signal, and silently counting to five or so before giving it
- using digit cards for all learners to show their answer at the same time
- asking learners to answer by writing their answers on small wipe boards.

The underlying teaching principle is to:

- allow time to discuss the various ways that learners reached the answer, to point out the range of possible strategies and use of jottings; help learners to recognise why one method is more efficient than another.

How does Higher Order Thinking fit with mental calculation?

Mental calculation is more than just recalling number facts. In itself, this is an important skill that aids learners to concentrate on their calculations, the problems and the methods involved. However, if they are to develop their ability in mental maths, they need to apply thinking skills to the problems presented to them and deepen their understanding of how to solve them. Bloom's Taxonomy offers one model for doing this from Remembering to Creating and this has been further developed below for six aspects of mathematics that involve mental calculation. They are supported with questions to exemplify what might be asked of learners to engage them in mental calculation activities and to stimulate discussion.

Recalling facts

- What is 3 add 7?
- What is 6 x 9?
- How many days are there in a week?... in four weeks?
- What fraction is equivalent to 0.25?
- How many minutes in an hour, in six hours?

Applying facts

- Tell me two numbers that have a difference of 12.
- If 3 x 8 is 24, what is 6 x 0.8?
- What is 20% of £30?
- What are the factors of 42?
- What is the remainder when 31 is divided by 4?

Hypothesising or predicting

- The number 6 is 1 + 2 + 3, the number 13 is 6 + 7. Which numbers to 20 are the sum of consecutive numbers?
- Roughly, what is 51 times 47?
- How many rectangles in the next diagram? And the next?
- On a 1 to 9 key pad, does each row, column and diagonal sum to a number that is a multiple of 3?

Designing and comparing procedures

- How might we count a pile of sticks?
- How could you subtract 37 from 82?
- How could we test a number to see if it is divisible by 6?
- How could we find 20% of a quantity?
- Are these all equivalent calculations: 34 − 19; 24 − 9; 45 − 30; 33 − 20; 30 − 15?

Interpreting results

- So what does that tell us about numbers that end in 5 or 0?
- Double 15 and double again; now divide your answer by 4. What do you notice? Will this always work?
- If 6 x 7 = 42 is 60 x 0.7 = 42?
- I know 5% of a length is 2 cm. What other percentages can we work out quickly?

Applying reasoning

- The seven coins in my purse total 23p. What could they be?
- In how many different ways can four learners sit at a round table?
- Why is the sum of two odd numbers always even?

Questions – Open? Closed? Fat? Thin?

The importance of the type of questions learners are presented with, when developing their mental maths skills, cannot be overestimated. Open and closed questions both have their place but we must be clear as to their purpose.

Closed
Help to establish specific areas of knowledge, skills and understanding. They often focus on learners providing explanations as to how and why something works and can be applied when identifying and developing approaches and strategies for a particular purpose.

Open
Help to generate a variety of alternative solutions and approaches that offer learners a chance to respond in different ways. They often focus on learners providing explanations and reasons for their choices and decisions and a comparison of which of the alternative answers are correct or why certain strategies are more efficient.

Closed Questions

Count these cubes.

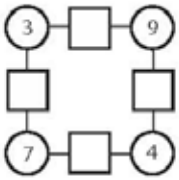
A chew costs 3p. A lolly costs 7p.
What do they cost altogether?

What is 6 – 4?

What is 2 + 6 – 3?

Is 16 an even number?

Write a number in each box so that it equals the sum of the two numbers on each side of it.



Copy and complete this addition table.

+	4	7
2		
6		

What are four threes?

What is 7 x 6?

How many centimetres are there in a metre?

Continue this sequence: 1, 2, 4...

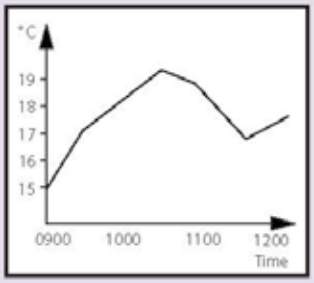
What is one-fifth add four-fifths?

What is 10% of 300?

What is this shape called?

This graph shows room temperature on 19 May.

What was the temperature at 10.00am?



Open Questions

How could we count these cubes?

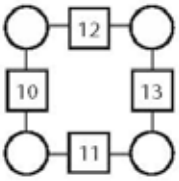
A chew and a lolly costs 10p altogether. What could each sweet cost?

Tell me two numbers with a difference of 2.

What numbers can you make with 2, 3 and 6?

How do you know whether 16 is even?

Write a number in each circle so that the number in each box equals the sum of the two numbers on each side of it. Find different ways of doing it.



Find different ways of completing this table.

	3	4
	7	

Tell me two numbers with a product of 12.

If 7 x 6 = 42, what else can you work out?

Tell me two lengths that together make 1 metre.

Find different ways of completing this sequence: 1, 2, 4...

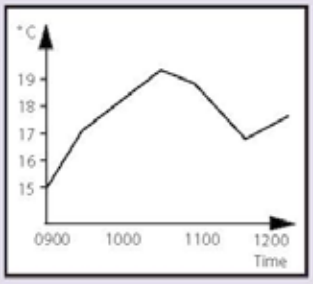
Write eight different ways of adding two numbers to make 1.

Find ways of completing: ...% of ... =30

Sketch some different triangles.

This graph shows room temperature on 19 May.

Can you explain it?



The use of questions is generally to:

Prompt thinking and establish a starting point.

Probe thinking and establish the confidence and security of the learners knowledge, skills and understanding.

Promote thinking to set a new challenge, problem or line of enquiry that learners can follow.

Hinge questions to extend learners thinking

Starting a piece of work:
How are you going to tackle this?
What information do you have?
What do you need to find out or do?
What operation(s) are you going to use?
Will you do it mentally, with a pencil and paper, using a number line, with a calculator..?
How are you going to record what you are doing?
What do you think the answer or result will be?
Can you estimate or predict?

Positive interventions to check progress:
Can you explain what you have done so far?
What else is there to do?
Why did you decide to use this method or do it this way?
Can you think of another method that might have worked?
Could there be a quicker way of doing this?
What did you mean by…?
What did you notice when…?
Why did you decide to organise your results like that?
Are you beginning to see a pattern or a rule?
Do you think that this would work with other numbers?

Support/guidance for learners:
Can you describe the problem in your own words?
Can you talk me through what you've done so far?
What did you do last time?
What is different this time?
Is there something that you already know that might help?
Could you try it with simpler numbers… fewer numbers… using a number line…?
What about putting things in order?
Would a table/picture/diagram/graph help?
Why not make a guess and check if it works?
Have you compared your work with anyone else's?

Assess learner's progress:
How did you get your answer?
Can you describe the method/pattern/rule to us all?
Can you explain why it works?
What could you try next?
Would it work with different numbers?
What if you had started with… rather than…?
What if you could only use…?
Is it a reasonable answer/result?
What have you learned or found out today?
If you were doing it again, what would you do differently?
What are the key points or ideas that you need to remember for the next lesson?

The underlying teaching principle here is to:

- support learners as problem solvers through developing their resilience in coming to a solution.

Lines of Progression

Addition and Subtraction – CfE Early Level

Recall: Learners should be able to derive and recall:	**Mental calculation skills:** Working mentally, with jottings if needed, learners should be able to:	**Mental methods or strategies:** Learners should understand when to and be able to apply these strategies:
• Number songs/rhymes/stories eg. 1, 2, 3, 4, 5 once I caught a fish alive, 1, 2 buckle my shoe, 10 little Indians. • Names of numerals to 10 • Conservation of number eg. knowing 3 is 3 regardless of arrangement of concrete materials / objects • Value of a set eg. counting 3 objects as 3. • Numbers in environment eg. signs around school… • Understand the language of daily routines. eg registration, lunches, birthdays • Understand signs or instructions eg 4 can play, one at a time, two at a time, with a partner. • Begin to use ordinal numbers in a given context eg 1st, 2nd, 3rd and last in a set for lining up, dates, sports	• Sort and create groups of objects by size, number or other properties. • Place/identify any given digit on a number line to 10 eg. before, after, in between • More/less comparison eg. a set of 5 and a set of 4 – which set has more/less?	• 1-1 correspondence when counting eg. matching games to encourage counting aloud. • Order numbers to 10 (forwards and backwards) • Use number lines to calculate 1 more/ less than • Share a group of items and discuss who has more/less.

• Use vocabulary such as bigger, smaller and 'the same' to compare groups of items. • Say number word sequences to at least 30 (forwards and backwards) • Recognise and use ordinal numbers up to 20th in context.	• Read and use a variety of number lines (straight, curved dials etc) • 1 more/less than • Count on and back from a given number, including bridging 10 • Combine sets of objects together and record this as a number sentence eg 1 + 5 = 6 • Addition facts within 5 eg. 1 + 0 = 1, 0 + 1 = 1… • Subtraction facts within 5 eg. 5 - 2 = 3 (inverse to addition facts.) • Explain answers in words, with materials, through drawings, on paper. • Use the signs +, -, =	• **Subitise** – Recognise a small number of objects without counting • Groupings within 5 eg 2 and 3, 4 and 1 • Groupings with 5 eg. 5 and 1, 5 and 2 • Groupings within 10 eg. 5 and 5, 4 and 6 • Partitioning a set to show Commutative Law - Understand that 3 + 4 is the same as 4 + 3 • Partitioning a set to show Associative Law – Understand that 6 + 3 + 7 is the same as 9 + 7

Addition and Subtraction – CfE First Level

Recall: Learners should be able to derive and recall:	**Mental calculation skills:** Working mentally, with jottings if needed, learners should be able to:	**Mental methods or strategies:** Learners should understand when to and be able to apply these strategies:
• number pairs with a total of 10, e.g. 3 + 7, or what to add to a single-digit number to make 10, e.g. 3 + ¨ = 10 • addition facts for totals to at least 5, e.g. 2 + 3, 4 + 3 • addition doubles for all numbers to at least 10, e.g. 8 + 8	• add or subtract a pair of single-digit numbers, e.g. 4 + 5, 8 - 3 • add or subtract a single-digit number to or from a teens number, e.g. 13 + 5, 17 – 3 • add or subtract a single-digit to or from 10, and add a multiple of 10 to a single-digit number, e.g. 10 + 7, 7 + 30 • add near doubles, e.g. 6 + 7 • count on from and back to zero in ones, twos, fives or tens e.g. count back in twos from 8	• reorder numbers when adding, e.g. put the larger number first • count on or back in ones, twos or tens • partition small numbers, e.g. 8 + 3 = 8 + 2 + 1 • partition and combine tens and ones • partition: double and adjust, e.g. 5 + 6 = 5 + 5 + 1 • use patterns of last digits, e.g. 0 and 5 when counting in fives
• addition and subtraction facts for all numbers up to at least 10, e.g. 3 + 4, 8 – 5 • number pairs with totals to 20 • all pairs of multiples of 10 with totals up to 100, e.g. 30 + 70, or 60 + ¨ = 100 • what must be added to any two-digit number to make the next multiple of 10, e.g. 52 + ¨ = 60 • addition doubles for all numbers to 20 and multiples of 10 to 50 e.g. 17 + 17, 40 + 40	• add or subtract a pair of single-digit numbers, including crossing 10, e.g. 5 + 8, 12 – 7 • add any single-digit number to or from a multiple of 10, e.g. 60 + 5 • subtract any single-digit number from a multiple of 10, e.g. 80 – 7 • add or subtract a single-digit number to or from a two-digit number, including crossing the tens boundary, e.g. 23 + 5, 57 – 3, then 28 + 5, 52 – 7 • add or subtract a multiple of 10 to or from any two-digit number, e.g. 27 + 60, 72 – 50 • add 9, 19, 29, … or 11, 21, 31, … • add near doubles, e.g. 13 + 14, 39 + 40	• reorder numbers when adding • partition: bridge through 10 and multiples of 10 when adding and subtracting • partition and combine multiples of tens and ones • use knowledge of pairs making 10, e.g.60 – 7 think 10 – 7 = 3, so 60 – 7 = 53 • partition: count on in tens and ones to find the total • partition: count on or back in tens and ones to find the difference • partition: add a multiple of 10 and adjust by 1 • partition: double and adjust

• addition and subtraction facts for all numbers to 20, e.g. 9 + 8, 17 – 9, drawing on knowledge of inverse operations • sums and differences of multiples of 10, e.g. 50 + 80, 120 – 90 • pairs of two-digit numbers with a total of 100, e.g. 32 + 68, or 32 + ¨ = 100 • addition doubles for multiples of 10 to 100, e.g. 90 + 90	• add and subtract groups of small numbers, e.g. 5 – 3 + 2 • add or subtract a two-digit number to or from a multiple of 10, e.g. 50 + 38, 90 – 27 • add and subtract two-digit numbers which do not bridge 10s or 100s e.g. 34 + 65, 68 – 35 • add near doubles, e.g. 18 + 16, 60 + 70	• reorder numbers when adding • identify pairs totalling 10 or multiples of 10 • partition: add tens and ones separately, then recombine • partition: count on in tens and ones to find the total • partition: count on or back in tens and ones to find the difference • partition: add or subtract 10 or 20 and adjust • partition: double and adjust • partition: count on or back in minutes and hours, bridging through 60 (analogue times)

Multiplication and Division – CfE First Level		
Recall: Learners should be able to derive and recall:	**Mental calculation skills:** Working mentally, with jottings if needed, learners should be able to:	**Mental methods or strategies:** Learners should understand when to and be able to apply these strategies:
• doubles of all numbers to 10, e.g. double 6 • odd and even numbers to 20	• count on from and back to zero in ones, twos, fives or tens	• use patterns of last digits, e.g. 0 and 5 when counting in fives
• doubles of all numbers to 20, e.g. double 13, and corresponding halves • doubles of multiples of 10 to 50, e.g. double 40, and corresponding halves • multiplication facts for the 2,4; 5 and 10 times-tables, and corresponding division facts • odd and even numbers to 100	• double any multiple of 5 up to 50, e.g. double 35 • halve any multiple of 10 up to 100, e.g. halve 90 • find half of even numbers to 40 • find the total number of objects when they are organised into groups of 2,4; 5 or 10	• partition: double the tens and ones separately, then recombine • use knowledge that halving is the inverse of doubling and that doubling is equivalent to multiplying by two • use knowledge of multiplication facts from the 2,4; 5 and 10 times-tables, e.g. recognise that there are 15 objects altogether because there are three groups of five
• multiplication facts to 10 x 10 and corresponding division facts • doubles of multiples of 10 to 100, e.g. double 90, and corresponding halves	• double any multiple of 5 up to 100, e.g. double 35 • halve any multiple of 10 up to 200, e.g. halve 170 • multiply one-digit or two-digit numbers by 10 or 100, e.g. 7 x 100, 46 x 10, 54 x 100 • find unit fractions of numbers and quantities e.g. $\frac{1}{8}$ of 24, $\frac{1}{7}$ of 35	• partition: when doubling, double the tens and ones separately, then recombine • partition: when halving, halve the tens and ones separately, then recombine • use knowledge that halving and doubling are inverse operations • recognise that finding a unit fraction is equivalent to dividing by the denominator and use knowledge of division facts • recognise that when multiplying by 10 or 100 the digits move one or two places to the left and zero is used as a place holder

Addition and Subtraction – CfE Second Level		
Recall: Learners should be able to derive and recall:	**Mental calculation skills:** Working mentally, with jottings if needed, learners should be able to:	**Mental methods or strategies:** Learners should understand when to and be able to apply these strategies:
• sums and differences of pairs of multiples of 10, 100 or 1000 • addition doubles of numbers 1 to 100, e.g. 38 + 38, and the corresponding halves • what must be added to any three-digit number to make the next multiple of 100, e.g. 521 + ¨ = 60 • pairs of fractions and decimal fractions that total 1	• add or subtract any pair of two-digit numbers, including crossing the tens and 100 boundary, e.g. 47 + 58, 91 – 35 • add or subtract a near multiple of 10, e.g. 56 + 29, 86 – 38 • add near doubles of two-digit numbers, e.g. 38 + 37 • add or subtract two-digit or three-digit multiples of 10, e.g. 120 – 40, 140 + 150, 370 – 180	• count on or back in hundreds, tens and ones • partition: add tens and ones separately, then recombine • partition: subtract tens and then ones, e.g. subtracting 27 by subtracting 20 then 7 • subtract by counting up from the smaller to the larger number • partition: add or subtract a multiple of 10 and adjust, e.g. 56 + 29 = 56 + 30 – 1, or 86 – 38 = 86 – 40 + 2 • partition: double and adjust • use knowledge of place value and related calculations, e.g. 140 + 150 = 290 using 14 + 15 = 29 • partition: count on or back in minutes and hours, bridging through 60 (analogue and digital times)

• sums and differences of decimals, e.g. 6.5 + 2.7, 7.8 – 1.3 • doubles and halves of decimals, e.g. half of 5.6, double 3.4 • what must be added to any four-digit number to make the next multiple of 1000, e.g. 4087 + ¨ = 5000 • what must be added to a decimal with units and tenths to make the next whole number, e.g. 7.2 + ¨ = 8	• add or subtract a pair of two-digit numbers or three-digit multiples of 10, e.g. 38 + 86, 620 – 380, 350+ 360 • add or subtract a near multiple of 10 or 100 to any two-digit or three-digit number, e.g. 235 + 198 • find the difference between near multiples of 100 or of 1000, e.g. 607 – 588, 6070 – 4087 • add or subtract any pairs of decimal fractions each with units and tenths, e.g. 5.7 + 2.5, 6.3 – 4.8	• count on or back in hundreds, tens, ones and tenths • partition: add hundreds, tens or ones separately, then recombine • subtract by counting up from the smaller to the larger number • add or subtract a multiple of 10 or 100 and adjust • partition: double and adjust • use knowledge of place value and related calculations, e.g. 6.3 – 4.8 using 63 – 48 • partition: count on or back in minutes and hours, bridging through 60 (analogue and digital times)
• addition and subtraction facts for multiples of 10 to 1000 and decimal numbers with one decimal place, e.g. 650 + ¨ = 930, ¨ – 1.4 = 2.5 • what must be added to a decimal with units, tenths and hundredths to make the next whole number, e.g. 7.26 + ¨ = 8	• add or subtract pairs of decimals with units, tenths or hundredths, e.g. 0.7 + 3.38 • find doubles of decimals each with units and tenths, e.g. 1.6 + 1.6 • add near doubles of decimals, e.g. 2.5 + 2.6 • add or subtract a decimal with units and tenths, that is nearly a whole number, e.g. 4.3 + 2.9, 6.5 – 3.8	• count on or back in hundreds, tens, ones, tenths and hundredths • use knowledge of place value and related calculations, e.g. 680 + 430, 6.8 + 4.3, 0.68 + 0.43 can all be worked out using the related calculation 68 + 43 • use knowledge of place value and of doubles of two-digit whole numbers • partition: double and adjust • partition: add or subtract a whole number and adjust, e.g. 4.3 + 2.9 = 4.3 + 3 – 0.1, 6.5 – 3.8 = 6.5 – 4 + 0.2 • partition: count on or back in minutes and hours, bridging through 60 (analogue and digital times, 12-hour and 24-hour clock)

Multiplication and Division – CfE Second Level		
Recall: Learners should be able to derive and recall:	**Mental calculation skills:** Working mentally, with jottings if needed, learners should be able to:	**Mental methods or strategies:** Learners should understand when to and be able to apply these strategies:
• with increasing speed and confidence, multiplication facts to 10 x 10 and the corresponding division facts • doubles of numbers 1 to 100, e.g. double 58, and corresponding halves • doubles of multiples of 10 and 100 and corresponding halves • fraction and decimal fraction equivalents of one-half, quarters, tenths and hundredths, e.g. $\frac{3}{10}$ is 0.3 and $\frac{3}{100}$ is 0.03 • factor pairs for known multiplication facts	• double any two-digit number, e.g. double 39 • double any multiple of 10 or 100, e.g. double 340, double 800, • halve the corresponding multiples of 10 and 100 eg half of 60, half of 400 • halve any even number to 200 • find unit fractions and simple non-unit fractions of numbers and quantities, e.g. $\frac{1}{7}$ of 21, $\frac{3}{8}$ of 24 • multiply and divide numbers to 1000 by 10 and then 100 (whole-number answers), e.g. 325 x 10, 42 x 100, 120 ÷ 10, 600 ÷ 100, 850 ÷ 10 • multiply a multiple of 10 to 100 by a single-digit number, e.g. 40 x 3 • multiply numbers to 20 by a single-digit, e.g. 17 x 3 • identify the remainder when dividing by 2, 5 or 10 • give the factor pair associated with a multiplication fact, e.g. identify that if 2 x 3 = 6 then 6 has the factor pair 2 and 3	• partition: double or halve the tens and ones separately, then recombine • use understanding that when a number is multiplied or divided by 10 or 100, its digits move one or two places to the left or the right and zero is used as a place holder • use knowledge of multiplication facts and place value, e.g. 7 x 8 = 56 to find 70 x 8, 7 x 80 • use partitioning and the distributive law to multiply, e.g.13 x 4 = (10 + 3) x 4 = (10 x 4) + (3 x 4) = 40 + 12 = 52

• squares to 10 x 10 • division facts corresponding to tables up to 10 x 10, and the related unit fractions, e.g. 7 x 9 = 63 so one-ninth of 63 is 7 and one-seventh of 63 is 9 • percentage equivalents of one-half, one-quarter, three-quarters, tenths and hundredths • factor pairs to 100	• multiply and divide two-digit numbers by 4 or 8, e.g. 26 x 4, 96 ÷ 8 • multiply two-digit numbers by 5 or 20, e.g. 32 x 5, 14 x 20 • multiply by 25 or 50, e.g. 48 x 25, 32 x 50 • double three-digit multiples of 10 to 500 and find the corresponding halves, e.g. 380 x 2, 760 ÷ 2 • find the remainder after dividing a two-digit number by a single-digit number, e.g. 27 ÷ 4 = 6 r 3 • multiply and divide whole numbers and decimal fractions by 10, 100 or 1000, e.g. 4·3 x 10, 0.75 x 100, 25 ÷ 10, 673 ÷ 100, 74 ÷ 100 • multiply pairs of multiples of 10 and a multiple of 100 by a single digit number, e.g. 60 x 30, 900 x 8 • divide a multiple of 10 by a single-digit number (whole number answers) e.g. 80 ÷ 4, 270 ÷ 3 • find fractions of whole numbers or quantities, e.g. $\frac{2}{3}$ of 27, $\frac{4}{5}$ of 70 kg • find 50%, 25% or 10% of whole numbers or quantities, e.g. 25% of 20 kg, 10% of £80 • find factor pairs for numbers to 100, e.g. 30 has the factor pairs 1 x 30, 2 x 15, 3 x 10 and 5 x 6	• multiply or divide by 4 or 8 by repeated doubling or halving • form an equivalent calculation, e.g. to multiply by 5, multiply by 10, then halve; to multiply by 20, double, then multiply by 10 • use knowledge of doubles/halves and understanding of place value, e.g. when multiplying by 50 multiply by 100 and divide by 2 • use knowledge of division facts, e.g. when carrying out a division to find a remainder • use understanding that when a number is multiplied or divided by 10 or 100, its digits move one or two places to the left or the right relative to the decimal point, and zero is used as a place holder • use knowledge of multiplication and division facts and understanding of place value, e.g. when calculating with multiples of 10 • use knowledge of equivalence between fractions and percentages, e.g. to find 50%, 25% and 10% • use knowledge of multiplication and division facts to find factor pairs

• squares to 12 x 12 • squares of the corresponding multiples of 10 • prime numbers less than 100 • equivalent fractions, decimal fractions and percentages for hundredths, e.g. 35% is equivalent to 0.35 or $\frac{35}{100}$ • more complex but commonly used equivalent fractions, decimal fractions and percentages. e.g. $33\frac{1}{3}$% is equivalent to $\frac{1}{3}$ or 0.33 66 $\frac{2}{3}$% is equivalent to $\frac{2}{3}$ or 0.67	• multiply pairs of two-digit and single-digit numbers, e.g. 28 x 3 • divide a two-digit number by a single-digit number, e.g. 68 ÷ 4 • divide by 25 or 50, e.g. 450 ÷ 25, 3200 ÷ 50 • double decimals with units and tenths and find the corresponding halves, e.g. double 7.6, half of 15.2 • multiply pairs of multiples of 10 and 100, e.g. 50 x 30, 600 x 20 • divide multiples of 100 by a multiple of 10 or 100 (whole number answers), e.g. 600 ÷ 20, 800 ÷ 400, 2100 ÷ 300 • multiply and divide two-digit decimal fractions such as 0.8 x 7, 4·8 ÷ 6 • find 10% or multiples of 10%, of whole numbers and quantities, e.g. 30% of 50 ml, 40% of £30, 70% of 200 g • simplify fractions by cancelling • scale up and down using known facts, e.g. given that three oranges cost 24p, find the cost of four oranges.	• form an equivalent calculation, e.g. to divide by 25, divide by 100, then multiply by 4; to divide by 50, divide by 100, then double • partition: use partitioning and the distributive law to divide 10s and units separately eg 92 ÷ 4 = (80 + 12) ÷ 4 = 20 + 3 = 23 • use knowledge of the equivalence between fractions and percentages and the relationship between fractions and division • recognise how to scale up or down using multiplication and division, e.g. if three oranges cost 24p: one orange costs 24 ÷ 3 = 8p four oranges cost 8 x4 = 32p

Addition and Subtraction – CfE Third Level		
Recall: Learners should be able to derive and recall:	**Mental calculation skills:** Working mentally, with jottings if needed, learners should be able to:	**Mental methods or strategies:** Learners should understand when to and be able to apply these strategies:
• create equivalent fractions • convert between whole and mixed numbers and fractions • round a decimal fraction to two decimal places • arrange integers in ascending or descending order of size • read scales and calculate differences involving integers e.g. temperature	• add and subtract commonly used fractions e.g. $\frac{1}{2} + \frac{1}{4}$, $\frac{3}{4} - \frac{1}{8}$, $1\frac{1}{3} - \frac{3}{4}$ • add or subtract any pairs of decimal fractions each with units and tenths, e.g. 15.7 + 22.5, 6.32 - 4.18, 7.45 - 3.88 • calculate the change e.g. £20 - £15.75 • add and subtract integers appropriately within a given context e.g. calculate the rise in temperature. or Peter has £12 in his bank. If he spends £15, what will his bank statement now show?	• create the appropriate equivalent fraction (common denominators), then add or subtract • partition: add the whole numbers and the decimal fractions separately, then recombine • partition: subtract the whole numbers and the decimal fractions separately, then recombine • partition: add or subtract a whole number and adjust, e.g. 7.45 - 3.88 = 7.45 - 4 + 0.12

Multiplication and Division – CfE Third Level

Recall: Learners should be able to derive and recall:	**Mental calculation skills:** Working mentally, with jottings if needed, learners should be able to:	**Mental methods or strategies:** Learners should understand when to and be able to apply these strategies:
• Prime numbers less than 100 and know that 1 is not a prime number • Cubed numbers to 5x5x5 and corresponding roots • Roots of perfect squares up to √144 and inverse of commonly used squared numbers e.g. √400 • Equivalent fractions, decimal fractions and percentages up to thousandths e.g. 3.5% = 0.035 = $\frac{35}{1000}$ $\frac{14}{200} = \frac{7}{100} = 7\% = 0.07$ • Simplify a ratio e.g. 8:24 = 1:3 • Convert from ratio to fraction e.g. the ratio of boys to girls is 2:3 therefore the fraction of boys in the class is $\frac{2}{5}$ • Order numbers which are in standard form	• Identify numbers with odd and even numbers of factors and no factor pairs other than 1 and itself • Multiple and divide whole numbers and decimal fractions by multiples of ten, hundred or a thousand, e.g. 4.3x 30, 630÷200 • Simplify fractions and ratios by recognising common factors and cancelling • Calculate percentages of whole numbers and quantities e.g. 11% of 40g, 35% of 500 • Scale up and down using known facts e.g. At present I get 200 text messages per month on my contract. If the cost of each text message doubles how many texts will I get per month? • Convert to/from standard form e.g. 1.255 x 107 = 12550000 or 7155000 = 7.155 x106 or 0.00036 = 3.6x10-4	• Use knowledge of multiplication and division facts to identify factor pairs and numbers with only two factors • Form an equivalent calculation e.g. 11% of 40 = 10% of 40 and 1% of 40 and recombine e.g. 35% of 500 = 10% of 500 x 7 ÷ 2 • Recognised how to scale up or down using multiplication and division e.g. If two builders take five days to build a wall how long will it take four builders to build the same wall? One builder 5 x 2 = 10 days Four builders 10 ÷ 5 = 2.5 days

The Strategies:
Taking a Closer Look

Mental Maths Strategies

Learners should learn number facts and multiplication tables 'by heart'. If they cannot recall these facts rapidly and always resort to basic counting strategy instead, they are distracted from thinking about the calculation strategy they are trying to use.

Addition and subtraction strategies

Features of addition and subtraction

Numbers can be added in any order. Take any pair of numbers, say 7 and 12, then:

7 + 12 = 12 + 7

This is the ***commutative law of addition***. It applies to addition but not subtraction.

In subtraction, order does matter. So, 5 – 3 is not the same as 3 – 5 but a series of consecutive subtractions can be taken in any order. For example 15 – 3 – 5 = 15 – 5 – 3

When three numbers are added together, they too can be taken in any order because of the ***associative law*** and the ***commutative law***. In practice, two of the numbers have to be added together or associated first, and then the third number is added to the associated pair to give the result of the calculation.

e.g. 7 + 5 + 3 = (7 + 5) + 3
= 7 + (5 + 3)
= (7 + 3) + 5

Because of the inverse relationship between addition and subtraction, every addition calculation can be linked with an equivalent subtraction calculation and vice versa.

For example the addition:

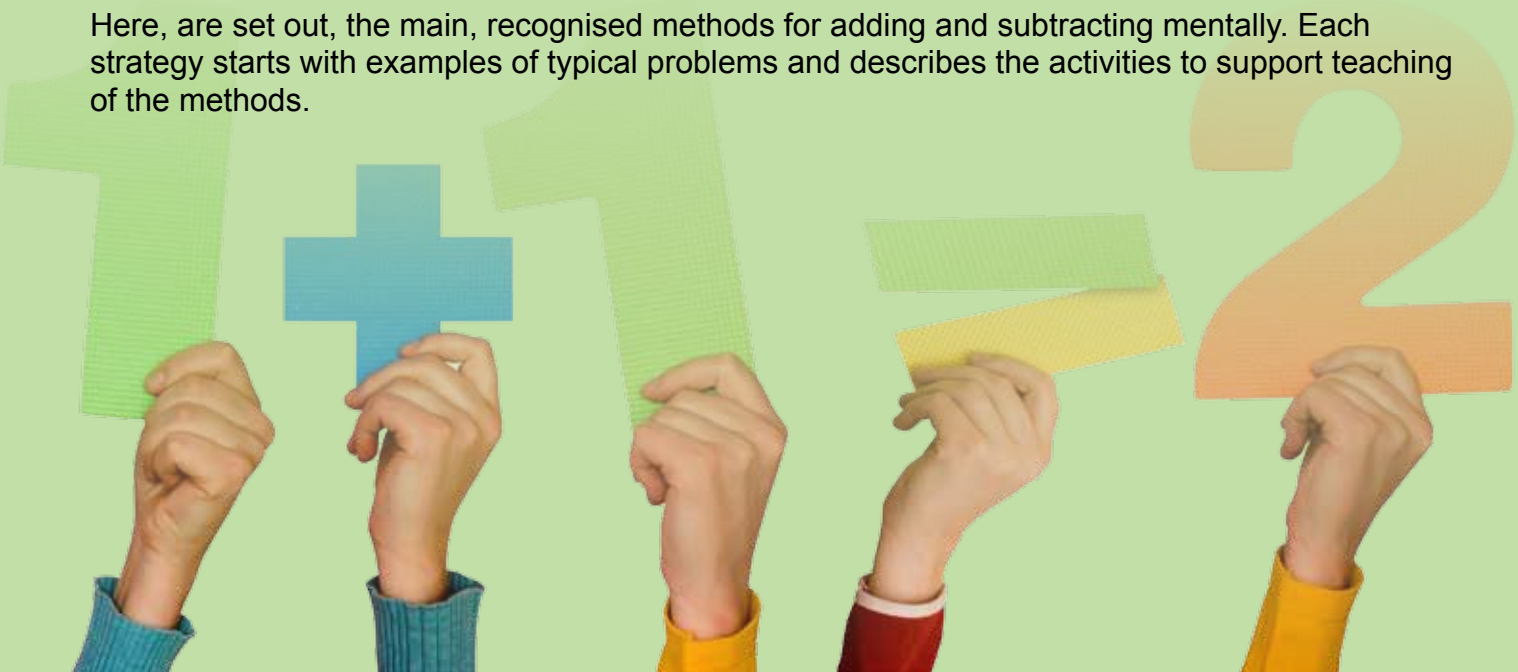
5 + 7 = 12 leads to 5 = 12 – 7 and 7 = 12 – 5

In the same way

13 – 6 = 7 leads to 13 = 7 + 6 and 6 = 13 – 7

Any numerical equivalence can be read from left to right or from right to left, so 6 + 3 = 9 can always be re-read as 9 = 6 + 3.

Here, are set out, the main, recognised methods for adding and subtracting mentally. Each strategy starts with examples of typical problems and describes the activities to support teaching of the methods.



The strategies covered are:

- Counting forwards and backwards.
- Reordering
- Partitioning – counting on or back
- Partitioning – bridging a multiple of 10
- Partitioning – compensating
- Partitioning – using 'near' doubles
- Partitioning – bridging through 60 to calculate a time interval

Counting forwards and backwards:

Children first meet counting by beginning at one and counting on in ones. Their sense of number is extended by beginning at different numbers and counting forwards and backwards in steps, not only of ones but also of twos, fives, tens, hundreds, tenths and so on. The image of a number line can help learners to appreciate the idea of counting forwards and backwards. They will also learn that, when adding two numbers together, it is generally easier to count on from the larger number rather than the smaller. Learners' 'counting on strategies' need to be regularly reviewed to encourage them to adopt more efficient methods as their learning progresses.

Exemplification

Progression	Example calculation	Possible counting strategy
	4 + 5	count on in ones from 4 (or in ones from 5)
	8 – 3	count back in ones from 8
	10 + 7	count on in ones from 10 (or use place value)
	13 + 5	count on in ones from 13
	17 – 3	count back in ones from 17
	18 – 6	count back in twos
	23 + 5	count on in ones from 23
	57 – 3	count back in ones from 57
	60 + 5	count on in ones or use place value
	80 – 7	count back in ones from 80 (or use knowledge of number fact to 10 and place value)
	27 + 60	count on in tens from 27
	72 – 50	count back in tens from 72
	50 + 38	count on in tens then in ones from 50
	90 – 27	count back in tens then in ones from 90
	34 + 65	count on in tens then ones from 34
	87 – 23	count back in tens then ones from 87
	35 + 15	count on in steps of 5 from 35
	73 – 68	count up from 68, counting 2 to 70 then 3 to 73
	47 + 58	count on 50 from 47, then 3 to 100 then 5 to 105
	124 – 47	count back 40 from 124, then 4 to 80, then 3 to 77
	570 + 300	count on in hundreds from 570
	960 – 500	count back in hundreds from 960
	3.2 + 0.6	count on in tenths
	1.7 + 0.55	Count on in tenths and hundredths

Activities:

1. Ask learners to count from zero in ones, one after the other round the class. When the adult with the group claps, they must count backwards. On the next clap they count forwards and so on.

   Extend to counting in twos, fives, threes and other multiples. Vary the starting number. Extend to negative numbers.

2. Ensure learners have experience of using bead strings to underpin their understanding of the number line.

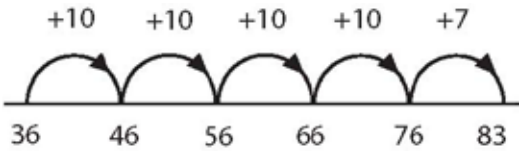
   Make a number line that goes up in tens, large enough for the whole class/group to see.

   Ask an individual learner to show where a number (e.g. 26) would fit on the line. Ask other learners to fit some numbers close to 26 (e.g. 23 or 28) on the line. They may need to adjust the positions of the numbers until they are satisfied with them. Get them to explain what they did to the rest of the class.

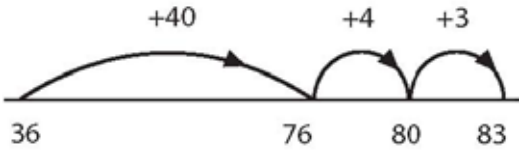
   This sort of activity encourages children to imagine where the numbers 1 to 9, 11 to 19 and 21 to 29 would appear on the line and to count on mentally before they decide where to place the number they are given. The idea can be extended to decimal fractions (e.g. with a line numbered 1, 2, 3 for positioning of numbers in tenths, or numbered 0.1, 0.2, 0.3 for positioning of numbers in hundredths.

3. Ask learners to record a two-digit number on an empty number line (e.g. 36 + 47)

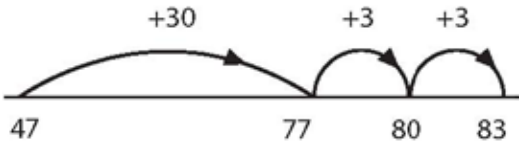
   (a) counting on from 36 initially in steps of 10



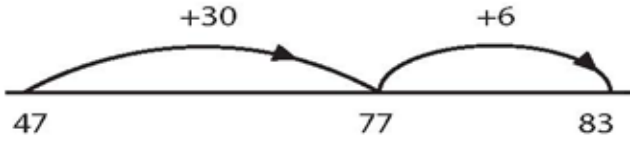
   (b) counting on a step of 40 to 76, then bridging through 80 using two steps.



   (c) counting on from 47, bridging through 80 using two steps.



(d) counting on 30 to 77, then using knowledge of number facts to 20 and place value



Empty or blank number lines or bead strings give learners a useful way to record their working and help you to see what method they are using. Discuss these different methods with the learners and encourage them to move to more efficient methods using fewer steps.

4. Move along an imaginary number line. Tell the learners which number you are standing on and what size steps you are taking. (e.g. I am standing on 15 and I am taking steps of 10). Invite them to visualise the number 15 on a number line and to tell you where you will be if you take one step forward (25). Take three more steps forward and ask, “Where am I now?” (55). Take two steps back and ask, “Where am I now?” (35) and so on.

   Activities such as these help learners visualise counting on or back. The activity can be used for larger numbers, visualising taking steps for ones, jumps for 10s and leaps for 100s. for example, tell them you are standing on 1570 and making leaps of 100 and ask them to visualise this, asking questions such as, “Where am I if I make two leaps forwards?”

Reordering:

Sometimes a calculation can be more easily worked out by changing the order of the numbers. The way in which learners rearrange numbers in a particular calculation will depend on which number facts they can recall or derive quickly.

It is important for learners to know when numbers can be reordered:

e.g. 2 + 5 + 8 = 8 + 2 + 5 or *15 + 8 – 5 = 15 – 5 + 8* or 23 – 9 – 3 = 23 – 3 – 9

and when they can’t:

e.g. 8 – 5 ≠ 5 – 8

The strategy of changing the order of numbers applies mainly when the question is written down. It is more difficult to reorder numbers if the question is presented orally.

Exemplification

Progression	Example calculation	Possible reordering strategy
	2 + 7	7 + 2
	5 + 13	13 + 5
	10 + 2 + 10	10 + 10 + 2
	5 + 34	34 + 5
	5 + 7 + 5	5 + 5 + 7
	23 + 54	54 + 23
	12 – 7 – 2	12 – 2 - 7
	13 + 21 + 13	13 + 13 + 21 (using double 13)
	6 + 13 + 4 + 3	6 + 4 + 13 + 3
	17 + 9 – 7	17 – 7 + 9
	28 + 75	75 + 28 (thinking of 28 as 25 + 3)
	12 + 17 + 8 + 3	12 + 8 + 17 + 3
	25 + 36 + 75	25 + 75 + 36
	58 + 47 - 38	58 – 38 + 47
	200 + 567	567 + 200
	3 + 8 + 7 + 6 + 2	3 + 7 + 8 + 2 + 6
	34 + 27 + 46	34 + 46 + 27
	180 + 650	650 + 180 (thinking of 180 as 150 + 30)
	1.7 + 2.8 + 0.3	1.7 + 0.3 + 2.8
	4.7 + 5.6 – 0.7	4.7 – 0.7 + 5.6 = 4 + 5.6

Activities:

1. Present learners with groups of three then four numbers that they can add mentally. Ensure that, in each group of numbers, there are two numbers that have a total of 10 (e.g. 8 + 3 + 2 + 5).
   Discuss their methods. See if any learners choose to add 8 + 2 first and then add on the 5 + 3, or linked the 3 + 5 and added 8 + (3 + 5) + 2.

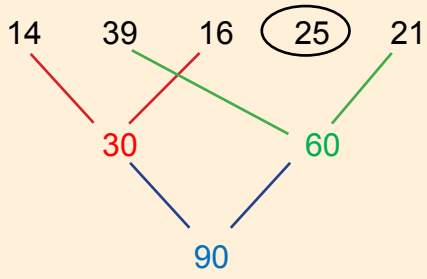
   Give learners similar examples and encourage them to look for pairs that add to make 10 or that make doubles before they start to add. Get them to make up similar examples for each other.

2. Have regular short practice sessions where learners are given 10 questions (e.g. 2 + 7 + 8 + 5 + 4 + 3) where some pairs total 10. Encourage learners to time their responses, keep a personal record of their times and try to beat their personal best (similar to Route 136).

Give learners the same set of questions at regular intervals and encourage them to see how rapidly they can get the answers. This should ensure that learners see that they have made progress.

3. When learners can find pairs of numbers that add to make multiples of 10, they can make use of this information when adding several numbers together.
   (e.g. 14 + 39 + 16 + 25 + 21)

   It is then sensible to pair numbers:



   90 + 25 = 115

   Learners should understand that it is sensible to look at all numbers that are to be added to see whether there are pairs that make convenient multiples of 10. The number tree (exemplified in the diagram above) can be a helpful model for pairing numbers.

4. In some sequences of numbers, the reordering strategy is useful and can give opportunities for an investigative approach.
   e.g. Find quick ways of finding these answers:

   1 + 2 + 3 + 4 + 5 + 6 = ?
   5 + 7 + 9 + 11 + 13 = ?
   3 + 6 + 9 + 12 + 15 + 18 = ?
   1 + 2 + 3 + 4 … + 98 + 99 = ?

   Series of numbers such as these are always easier to add by matching numbers in pairs. In the first example, it is easier to add 1 + 6 = 7, 2 + 5 = 7, 3 + 4 = 7 and then to find 7 x 3. In the last example combining 1 + 99, 2 + 98 and so on up to 49 + 51 gives 100 x 49 + 50 = 4950. From this, it may be worthwhile for learners to investigate if they can come up with a mathematical formula for calculating the answer to any list of numbers.

5. Use a set of number cards, making sure that there are pairs that make multiples of 10. Divide the class into groups of three and give each learner a card. Ask each group to add their numbers together. Encourage them to look for pairs of numbers to link together. List all the totals on a whiteboard (interactive?) and ask whose numbers give the largest total. Have learners swap their cards with another group and repeat. Cards should be organised so that each group of learners get cards that match their skills in number.

   The activity can be extended to decimal fractions with the aim of making pairs that make a whole number (e.g. 1.4, 3.2, 0.6, 0.2, 1.6, 0.8, 2.3)

Partitioning: counting on or back:

It is important for learners to know that numbers can be partitioned into, for example, hundreds, tens and ones, so that 326 = 300 + 20 + 6. In this way, numbers are seen as wholes, rather than as a collection of single digits in columns.

This way of partitioning numbers can be a useful strategy for adding and subtracting pairs of numbers. Both numbers can be partitioned, although it is often helpful to keep the first number as it is and to partition just the second number.

Exemplification

Progression	Example calculation	Possible counting on or back strategy
	30 + 47	30 + 40 + 7
	78 – 40	70 + 8 – 40 = 70 – 40 + 8
	17 + 14	10 + 7 + 10 + 4 = 10 + 10 + 7 + 4
	23 + 45	40 + 5 + 20 + 3 = 40 + 20 + 5 + 3
	68 – 32	60 + 8 – 30 – 2 = 60 – 30 + 8 - 2
	55 + 37	50 + 5 + 30 + 7 = 85 + 7
	365 – 40	300 + 60 + 5 – 40 = 300 + 60 – 40 + 5
	43 + 28 + 51	40 + 3 + 20 + 8 + 50 + 1 = 40 + 20 + 50 + 3 + 8 + 1
	5.6 + 3.7	5.6 + 3 + 0.7 = 8.6 + 0.7
	4.7 – 3.5	4.7 – 3 – 0.5
	540 + 280	540 + 200 + 80
	276 – 153	276 – 100 – 50 – 3

Activities:

1. Use a dice marked: 1, 1, 10, 10, 100, 100 for the game 'Target 500' with a group of learners. Each player may roll it as many times as they wish, adding the score from each roll and aiming at the target of 500. They must not 'overshoot'. If they do, they go bust!

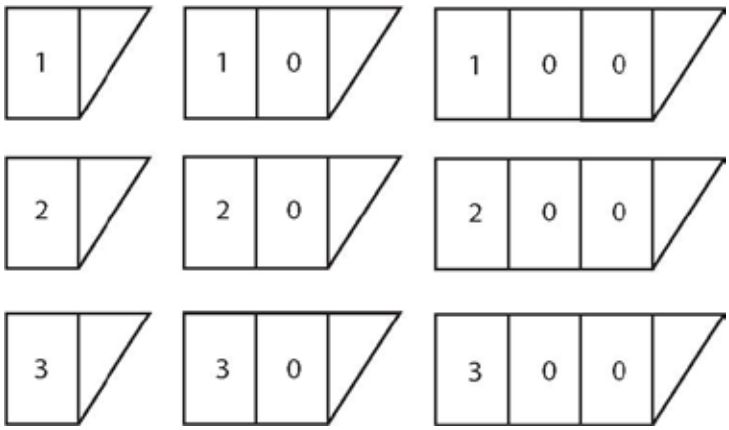
   For example, a sequence of rolls may be:

   10, 10, 1, 100, 1, 100, 100, 1, 1, 10, 100.

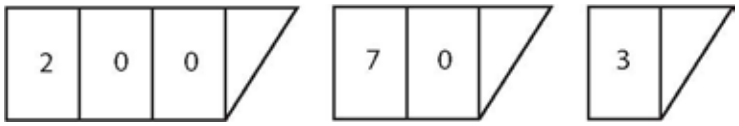
   At this point, with a total of 434, a player might decide not to risk another roll (in case 100 is rolled) and stop, or to hope for another 10. The winner is the player who gets nearest to 500.

   This game practises building up numbers by mental addition using ones, tens and hundreds.

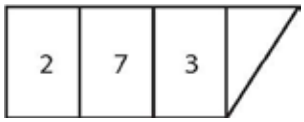
2. Use place value cards 1 to 9, 10 to 90 and 100 to 900, for example:



Ask children to use the cards to make a two-digit or a three-digit number by selecting the cards and placing them on top of each other.



For example, to make 273, the cards can be placed over each other to make:



This could be used as a class activity, in which case individual learners could select the appropriate card and put it in the correct place. Alternatively, it could be used with individuals as a diagnostic task to check whether or not they understand place value in this context.

3. With a group of three to five players, play a game using the place value cards 1 to 9, 10 to 90 and 100 to 900. You also need a set of two-digit and three-digit number cards to use as target numbers,

   e.g.

153	307	682	914	530	890
745	201	26	79	468	96

Put the target numbers in a pile and turn them over, one by one, to set a target. Deal the 27 place value cards between the players.

Player A inspects their cards to see if they have any part of the target number. If so, they put it on the table.

Play continues anti-clockwise. Player B checks to see whether they have another part of the number, followed by players C, D, and so on. Whoever completes the target number keeps it.

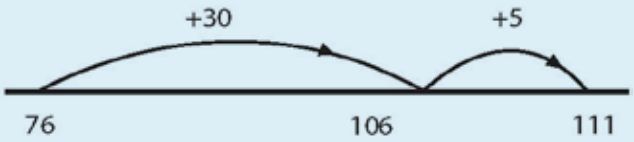
The winner is the player who wins the most target numbers.

Games like this motivate practice in partitioning numbers into hundreds, tens and units.

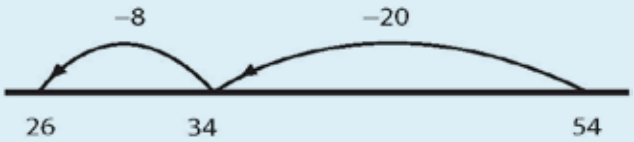
4. Use the empty number line to add or subtract two-digit numbers by partitioning the second number and counting on or back in tens then ones,

   e.g.

   76 + 35:



   54 – 28:

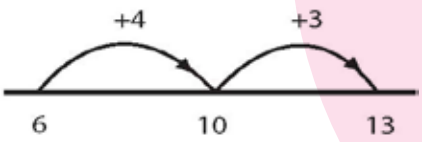


Empty number lines are a useful way to record how learners use multiples of 10 or 100 to add or subtract. They give a means for discussing different methods and encourage the use of more efficient methods.

Partitioning: bridging through multiples of 10:

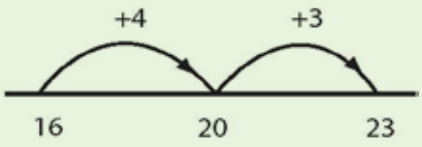
An important aspect of having an appreciation of number is to know how close a number is to the next or the previous multiple of 10: to recognise, for example, that 47 is 3 away from 50, or that 47 is 7 away from 40.

In mental addition or subtraction, it is often useful to count on or back in two steps, bridging a multiple of 10. The empty number line, with multiples of 10 as 'landmarks', is helpful, since learners can visualise jumping to them. For example, 6 + 7 is worked out in two jumps, first to 10, then to 13. The answer is the last point marked on the line, 13.



Subtraction, the inverse of addition, can be worked out by counting back from the larger number. But it can also be represented as the difference or 'distance' between two numbers. The distance is often found by counting up from the smaller to the larger number, again bridging through multiples of 10 or 100. This method of complementary addition is called 'shopkeeper's method' because it is like a shop assistant counting out change. So the change from £1 for a purchase of 37p is found by counting coins into the hand: '37p and 3p is 40p, and 10p makes 50p, and 50p makes £1'.

The empty number line can give an image for this method. The calculation 23 – 16 can be built up as an addition:



'16 and 4 is 20, and 3 is 23, so add 4 + 3 for the answer.' In this case the answer of 7 is not a point on the line but is the total distance between the two numbers 16 and 23.

A similar method can be applied to decimal fractions, but here, instead of building up to a multiple of 10, bridging is through the next whole number.

So 2.8 + 1.6 is 2.8 + 0.2 + 1.4 = 3 + 1.4.

Exemplification

Progression	Example calculation	Possible bridging through ten strategy
	5 + 8 or 12 – 7	5 + 5 + 3 or 12 – 2 – 5
	65 + 7 or 43 – 6	65 + 5 + 2 or 43 – 3 – 3
	24 – 19	19 + 1 + 4
	49 + 32	49 + 1 + 31
	90 – 27	27 + 3 + 60
	57 + 34 or 92 – 25	57 + 3 + 31 or 92 – 2 – 20 – 3
	84 – 35	35 + 5 + 40 + 4
	607 – 288	288 + 12 + 300 + 7
	6070 – 4987	4987 + 13 + 1000 + 70
	1.4 + 1.7 or 5.6 – 3.7	1.4 + 0.6 + 1.1 or 5.6 – 0.6 – 3 – 0.1
	0.8 + 0.35	0.8 + 0.2 + 0.15
	8.3 – 2.8	2.8 + 0.2 + 5.3 or 8.3 – 2.3 – 0.5

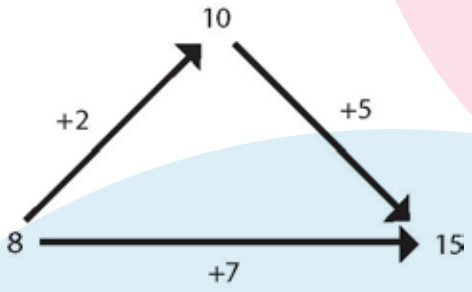
Activities:

1. Show the class a single-digit number and ask a learner to find its complement to 10. Repeat this many times, encouraging learners to respond as quickly as they can. Then offer two-digit numbers and ask for complements to 100. Extend to decimals. What must be added to 0.78 to make 1?

   Activities such as these give practice so that learners can acquire rapid recall of complements to 10 or 100 or 1. Using practical resources, such as multilink cubes or hundred squares helps learners to visualise complements.

2. Give the class two single-digit numbers to add. Starting with the first number (the larger), ask what part of the second number needs to be added to make 10, then how much more of it remains to be added on. Show this on a diagram like this:

   Example 8 + 7 = (8 + 2) + 5 = 10 + 5 = 15



   Diagrams like this are a useful model for recording how the starting number is built up to 10 and what remains to be added. Learners can be given blank diagrams and asked to use them to add sets of numbers less than 10.

3. Give examples based on money, asking children to show what coins they would use to build up to the next convenient amount. For example:
   - A packet of crisps costs 27p. How much change do you get from 50p? 27p and 3p is 30p. Another 20p makes it up to 50p. The change is 3p + 20p = 23p.
   - A paperback book costs £4.87. How much change do you get from £10?

   This is natural way to find a difference when using money. Coins provide a record of how the change was given. This method can usefully be applied to other instances of subtraction, such as "The cake went into the oven at 4.35pm. It came out at 5.15pm. How long did it take to cook?"

4. Write a set of decimals on the board, such as:

   3.6, 1.7, 2.4, 6.5, 2.3, 1.1, 1.5, 1.8, 2.2, 3.9

   Ask learners to find pairs that make a whole number.

   Extend the activity to finding pairs of numbers that make a whole number of tenths,

   e.g. 0.07, 0.06, 0.03, 0.05, 0.04, 0.05, 0.09, 0.01

   The first activity with decimals is to build up to whole numbers, so 3.6 is added to 2.4 to make 6. In the case of hundredths, pairs of numbers such as 0.06 and 0.04 can be added to make 0.1

Partitioning: compensation:

This strategy is useful for adding and subtracting numbers that are close to a multiple of 10, such as numbers that end in 1 or 2, or 8 or 9. The number to be added or subtracted is rounded to a multiple of 10 plus or minus a small number. For example, adding 9 is carried out by adding 10, then subtracting 1; subtracting 18 is carried out by subtracting 20, then adding 2.

A similar strategy works for adding or subtracting decimals that are close to whole numbers. For example:

1.4 + 2.9 = 1.4 + 3 – 0.1 or 2.45 – 1.9 = 2.45 – 2 + 0.1.

Exemplification

Progression	Example calculation	Possible compensating strategy
	34 + 9	34 + 10 – 1
	34 + 19	34 + 20 – 1
	34 + 29 and so on	34 + 30 – 1 and so on
	34 + 11	34 + 10 + 1
	34 + 21	34 + 20 + 1
	34 + 31 and so on	34 + 30 +1 and so on
	70 – 9	70 – 10 + 1
	53 + 12	53 + 10 + 2
	53 – 12	53 – 10 – 2
	53 + 18	53 + 20 – 2
	84 – 18	84 – 20 + 2
	38 + 68	38 + 70 – 2
	95 – 78	95 – 80 + 2
	58 + 32	58 + 30 + 2
	64 – 32	64 – 30 – 2
	138 + 69	138 + 70 – 1
	405 – 399	405 – 400 + 1
	2½ + 1¾	2½ + 2 - ¼
	5.7 + 3.9	5.7 + 4.0 – 0.1
	6.8 – 4.9	6.8 – 5.0 + 0.1

Activities:

1. Prepare two sets of cards for a subtraction game. Set A has numbers from 12 to 27. Set B contains only 9 and 11 so that the game involves subtracting 9 and 11. Shuffle the cards and place them face down.

   Each child needs a playing board.

15	3	9	16
5	18	4	17
13	7	12	8
6	11	14	10
18	1	2	17

   Learners take turns to choose a number from set A and then one from set B.

   They subtract the number from set B from the one from set A and mark the answer on their board. The first person to get three numbers in a row on their board wins.

   Discuss the strategy used to subtract. Encourage the use of compensating.

   By changing the numbers in set A, negative numbers can also be used.

2. Use a number square for adding tens and numbers close to 10. To find 36 + 28, first find 36 + 30 by going down three rows, then compensate by going back along that row two places:

31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70

   Learners need to know that they use the table to add 10 to any number by moving down to the number below it. For example 36 + 10 = 46, which is just below 36, and 36 + 20 is 56, to be found two rows below. Subtracting 10 is modelled by moving to numbers in the row above. Learners can use this strategy for adding or subtracting numbers that are close to a multiple of 10 by finding the correct row and then moving to the right or the left.

3. Practise examples of subtracting multiples of 10 or 100 such as 264 – 50,

   857 – 300.

   Then ask learners to subtract numbers such as 49 or 299, and so on. Encourage them to use rounding and compensating, so:

   264 – 49 = 264 – 50 + 1 and

   857 – 299 = 857 – 300 + 1.

   Slowly move on to numbers that are further away from a multiple of 10 or 100, such as 7 or 92.

   This activity could be a class activity. Individual learners round the class could give the answer. Encourage explanation of strategies.

Partitioning: using near doubles:

If learners have instant recall of doubles, they can use this information when adding two numbers that are very close to each other. So, knowing that 6 + 6 = 12, they can be encouraged to use this to help them find 7 + 6 rather than use a counting on strategy or bridging through 10.

Exemplification

Progression	Example calculation	Possible compensating strategy
	6 + 7	is double 6 and add 1 or double 7 and subtract 1
	13 + 14	is double 13 and add 1 or double 14 and subtract 1
	39 + 40	is double 40 and subtract 1
	18 + 16	is double 18 and subtract 2 or double 16 and add 2
	60 + 70	is double 60 and add 10 or double 70 and subtract 10
	76 + 75	is double 76 and subtract 1 or double 75 and add 1
	160 +170	is double 150 then add 10, then add 20 or double 160 and add 10 or double 170 and subtract 10
	2.5 + 2.6	is double 2.5 and add 0.1 or double 2.6 and subtract 0.1

Activities:

1. Choose a 'double fact' and display it on the board, for example:

   8 + 8 = 16

   Invite someone to give the total. Then ask for suggestions of addition facts that learners can make by changing one of the numbers, for example:

   8 + 9 = 17 7 + 8 = 15

   Extend the activity by giving learners double facts that they might not know, such as:

   17 + 17 = 34, 28 + 28 = 56, 136 + 136 = 272

   Then ask learners to say how they could work out:

   17 + 18 or *16 + 17* or 27 + 28 or *136 + 137*.

   Invite learners to give their own double fact and ask other learners to suggest some addition facts that they can generate from it. This activity can be extended to decimal fractions.

2. Play ‘Think of a number’. Use a rule that involves doubling and adding or subtracting a small number, for example:

   ‘I’m thinking of a number.

   I doubled it and added 3.

   My answer is 43.

   What was my number?’

   ‘Think of a number’ activities require learners to ‘undo’ a process by using inverse operations. This activity gives practice in both halving and doubling. Invite learners to invent similar examples themselves.

3. Ask learners to practise adding consecutive numbers such as 45 and 46. Then give learners statements such as: “I add two consecutive numbers and the total is 63.” Ask them: “Which numbers did I add?”

   Knowing doubles of numbers is useful for finding the sum of consecutive numbers. The reverse process is more demanding.

Partitioning: bridging through 60 to calculate a time interval:

Time is a universal non-metric measure.

A digital clock displaying 9.59 will, in two minutes time, read 10.01 not 9.61. When learners use minutes and hours to calculate time intervals, they have to bridge through 60.

So to find the time 20 minutes after 8.50am, for example, learners might say 8.50am plus 10 minutes takes us to 9.00am, then add another 10 minutes.

Exemplification

Progression	Example calculations
	It is 10.30am. How many minutes to 10.45am?
	It is 3.45pm. How many minutes to 4.15pm?
	I get up 40 minutes after 6.30am. What time is that?
	What is the time 50 minutes before 1.10pm?
	It is 4.25pm. How many minutes to 5.05pm?
	What time will it be 26 minutes after 3.30am?
	What was the time 33 minutes before 2.15pm?
	It is 4.18pm. How many minutes to 5.00pm? 5.26pm?
	It is 08.35. How many minutes is it to 09.15?
	It is 11.45. How many hours and minutes is it to 15.20?
	A train leaves Dumfries for Dundee at 22.33. The journey takes 2 hours 47 minutes. What time does the train arrive?

Activities:

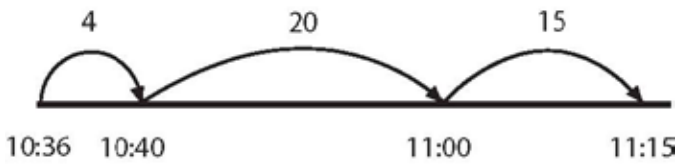
1. Have a digital clock in the classroom.

Ask the class to look at it at various times of the day and ask: 'How many minutes is it to the next hour (or next o'clock)?'

Encourage children to count on from 36 to 40, then to 50, then to 60, to give a total of 24 minutes.

Then ask questions such as: 'How long will it be to 11.15?' Get them to count on to 11.00 and then add on the extra 15 minutes.

The calculation can be modelled on a number line labelled in hours.



Some learners may think that minutes on digital clocks behave like ordinary numbers so that they might count on 59, 60, 61 and so on, not realising that at 60 the numbers revert to zero as the hour is reached. Draw attention to what happens to the clock soon after, say, 07.59 and stress the difference between this and other digital meters such as electricity meters or the meters that give the distance travelled by a bicycle or car.

2. Use local bus or train timetables (e.g. Stranraer to Dumfries).

Stranraer (Depart)	Dumfries (Arrive)
08.30	10.05
10.13	11.42
12.30	13.58
15.19	16.48

Ask questions such as “How long does the 08.30 bus take to get to Dumfries?” Encourage learners to count up to 09.00 and then to add on the extra 45 minutes.

Ask: “Which bus takes the shortest/longest time?”

Suggest that learners build the starting times up to the next hour and then add on the remaining minutes.

3. Plan a journey using information from a timetable. For example, coaches arrive and leave Gretna at these times:

	Arrive	**Leave**
Coach A	08.00	14.30
Coach B	09.30	15.45
Coach C	10.15	16.00
Coach D	11.45	17.30

Ask questions such as: ‘Which coach gives you the most/least time in Gretna?’

Discuss the strategies that learners use to find these times.

*For **Coach C**, for example, some might bridge 10.15 up to 11.00 and then find the number of remaining hours; others might bridge from 10.15 through 12.15 to 15.15, counting in hours and then add on the remaining 45 minutes.*
Learners need to remember that they need to count the hours 10, 11, 12 and then start again with 1, 2, and so on.

Again, an empty number line can be used to model any calculations.

Multiplication and division strategies

Features of multiplication and division

Three different ways of thinking about multiplication are:

- as repeated addition, for example 3 + 3 + 3 + 3
- as an array, for example four rows of three objects
- as a scaling factor, for example, making a line 3 cm long four times as long.

The use of the multiplication sign can cause difficulties. 3 x 4, read as “3 multiplied by 4” will mean four lots of three (3 + 3 + 3 + 3). However, when read as “three times four” it means three lots of four (4 + 4 + 4).

Fortunately, multiplication is commutative: 3 x 4 is equal to 4 x 3, so the outcome is the same. It is also a good idea to encourage learners to think of any product either way round, as 3 x 4 or as 4 x 3, as this reduces the facts that they need to remember through the reinforcement of the commutative law.

A useful link between multiplication and addition allows learners to work out new facts from facts that they already know. For example, the learner who can work out the answer to 6 x 8 (six eights) by recalling 5 x 8 (five eights) and then adding 8 will, through regular use of this strategy, become more familiar with the fact that 6 x 8 is 48.

Another feature of multiplication occurs in an expression such as (4 + 5) x 3, which involves both multiplication and addition. The distributive law of multiplication over addition means that:

(4 + 5) x 3 = (4 x 3) + (5 x 3)

This feature can be very useful in mental calculations.

Division and multiplication are inverse operations. Every multiplication calculation can be linked to an equivalent division calculation and vice versa. This link means that if learners learn multiplication facts so that they can recall them almost instantly they should also be able to recall quickly the corresponding division facts.

Because good knowledge of multiplication facts underpins all other multiplication and division calculations, written and mental, it is important that learners commit the multiplication tables to 10 x 10 to memory and derive corresponding division facts by the end of 1st Level, building up speed and accuracy through 2nd Level. They can also develop their skills at multiplying and dividing a range of whole numbers and decimal fractions during 2nd Level.

Here, are set out, the main, recognised methods for multiplying and dividing mentally. Each strategy starts with examples of typical problems and describes the activities to support teaching of the methods.

The strategies covered are:

- Knowing multiplication and division facts to 10 x 10
- Doubling and halving
- Multiplying and dividing by multiples of 10
- Multiplying and dividing by single-digit numbers and multiplying by two-digit numbers
- Finding fractions, decimal fractions and percentages

Multiplication and division facts to 10 x 10:

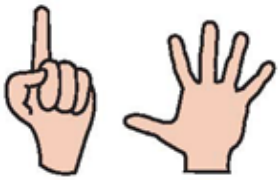
Fluent recall of multiplication and division facts relies on regular opportunities for practice. Generally, frequent short sessions are more effective than longer, less frequent sessions. It is crucial that the practice involves as wide a variety of activities, situations, questions and language as possible and that it leads to deriving and recognising number properties, such as doubles and halves, odd and even numbers, multiples, factors and primes.

Exemplification

Progression	Expectations of learners with examples
	Count on, from and back to zero in ones, twos, fives and tens.
	Recognise odd and even numbers to 20
	Recall the doubles of all numbers to 10
	Derive and recall doubles of all numbers to 20, and doubles of multiples of 10 to 50, and corresponding halves.
	Derive and recall multiplication facts for 2, 4; 5 and 10 times tables and corresponding division facts.
	Recognise odd and even numbers to 100
	Recognise multiples of 2, 4; 5 and 10
	Derive and recall doubles multiples of 10 to 100 and corresponding halves.
	Derive and recall multiplication facts for the 2, 4; 5, 10; 3 and 6 times tables and corresponding division facts.
	Recognise multiples of 2, 4; 5, 10; 3 and 6 up to the tenth multiple.
	Identify doubles of two digit numbers and corresponding halves.
	Derive doubles of multiples of 10 and 100 and corresponding halves.
	Derive and recall multiplication facts up to 10 x 10 and corresponding division facts.
	Recognise multiples of 2, 4, 8; 5, 10; 3, 6, 9; and 7 up to the tenth multiple.
	Recall squares of numbers to 10 x 10
	Use multiplication facts to derive products of pairs of multiples of 10 and 100 and corresponding division facts.
	Recall squares of numbers to 12 x 12 and derive corresponding squares of multiples of 10.
	Use place value and multiplication facts to derive related multiplication and division fact involving decimal fractions (e.g. 0.8 x 7, 4.8 ÷ 6)
	Identify factor pairs of two digit numbers.
	Identify prime numbers less than 100

Activities:

1. Learners who are able to count in twos, fives and tens can use this knowledge to work out other facts such as 2 x 6, 5 x 4, 10 x 9. Show the learners how to hold out their fingers and count, touching each finger in turn. So for 6 x 2 (six twos), hold up 6 fingers:



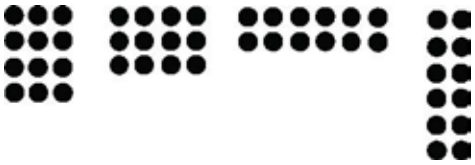
As learners touch each of the six fingers in turn, they say '2, 4, 6, 8, 10, 12' to get the answer 12.

For 4 x 5 (four fives), hold up four fingers:

This time they say '5, 10, 15, 20' to get 20.

Touching the fingers in turn is a means of keeping track of how far the children have gone in creating a sequence of numbers. The physical action can later be visualised without any actual movement.

2. Discuss ways of grouping dots in a rectangular array (single-line arrays are not allowed in this game), e.g. 12 dots can be represented as:



Play this game. Player A takes a handful of counters, counts them and tells player B how many there are. Player B then says how the counters can be arranged in a rectangular array and proceeds to make it. If both players agree the array is correct, player B gets a point.

Both players record the two multiplication facts that the array represents.

e.g. a 5 by 3 array is recorded as 15 = 3 x 5 and 15 = 5 x 3, or 3 x 5 = 15 and 5 x 3 = 15.

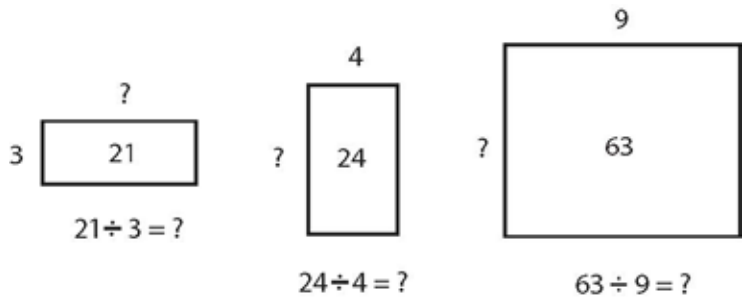
After the game discuss numbers that can ***only*** be made into a single row or single column array (*THE PRIME NUMBERS*).

*Arranging counters in a rectangular array is a helpful introduction to understanding about factors. If a number can be arranged in a rectangle (excluding a straight line), then it can be factorised. Numbers that can **only** be arranged as a straight line are **primes**.*

3. Learners can develop new facts by partitioning one of the numbers. So, for example, e.g. 12 x 7 = (10 + 2) x 7 = 10 x 7 + 2 x 7 = 84
   Subtraction can be used similarly, so ‘nine eights are ten eights minus one eight’.
   Another strategy is to use factors, so 7 x 6 is seen as 7 x 3 x 2.

   These strategies are useful when new tables are being developed.

4. Use rectangles to practise division:



   This area model helps to show that multiplication and division are inverse operations.

Doubling and halving:

The ability to double numbers is useful for multiplication. Historically, multiplication was carried out by a process of doubling and adding. Most people find doubles the easiest multiplication facts to remember, and they can be used to simplify other calculations.

Sometimes it can be helpful to halve one of the numbers in a multiplication calculation and double the other.

Exemplification

Progression	Expectations of learners with examples
	Double all numbers to 10 e.g. double 9
	Double all numbers to 20 and find the corresponding halves e.g. double 7, half of 14
	Double multiples of 10 to 50 e.g. double 40 and find corresponding halves.
	Double multiples of 5 to 50 and find the corresponding halves e.g. double 35, half of 70.
	Double multiples of 10 to 100 e.g. double 90 and corresponding halves
	Double multiples of 5 to 100 and find the corresponding halves e.g. double 85, halve 170
	Double any two digit number and find the corresponding halves e.g. double 47 , half of 94
	Double multiples of 10 and 100 and find the corresponding halves e.g. double 800, double 340, half of 1600, half of 680
	Form equivalent calculations and use doubling and halving such as: • Multiply by 4 by doubling twice e.g. 16 x 4: 16 x 2 = 32, 32 x 2 = 64 • Multiply by 8 by doubling three times e.g. 12 x 8: 12 x 2 = 24, 24 x 2 = 48, 48 x 2 = 96 • Divide by 4 by halving twice e.g. 104 ÷ 4: 104 ÷ 2 = 52, 52 ÷ 2 = 26 • Divide by 8 by halving three times e.g. 104 ÷ 8: 104 ÷ 2 = 52, 52 ÷ 2 = 26, 26 ÷ 2 = 13 • Multiply by 20 by doubling then multiplying by 10 e.g. 53 x 20: 53 x 2 = 106, 106 x 10 = 1060
	Multiply by 50 by multiplying by 100 and halving
	Multiply by 25 by multiplying by 100 and halving twice
	Double decimal fractions with units and tenths e.g. double 7.6 and find the corresponding halves e.g. half of 15.2
	Form equivalent calculations and use doubling and halving such as: • Divide by 25 by dividing by 100 then multiplying by 4 e.g. 460 ÷ 25 = 4.6 x 4 = 18.4 • Divide by 50 by dividing by 100 then doubling e.g. 270 ÷ 50 = 2.7 x 2 = 5.4

Activities:

1. Play 'Doubles dominoes' with a group of learners: this needs a set of dominoes in which, for example, 7 x 2, 2 x 7, 7 + 7 and 14 can be matched.

Record/observe which facts learners can recall quickly.

2. Ask learners to halve a two-digit number such as 56. Discuss the ways in which they might work it out. Show them that, unless they 'know', it may be better to partition it as 50 + 6 and to work out half of 50 and half of 6, then add these together.

   Ask a learner to suggest an even two-digit number and challenge others to find a way of halving it.

   Some learners may be able to halve an odd number, for example 47, by saying that it is 23 and a half.

   If learners are familiar with the halves of the multiples of 10 beyond 100, they can partition three-digit numbers in order to halve them.
   e.g. to halve 364 = half of (300 + 60 +4) = 150 + 30 + 2

3. Investigate doubling and halving number chains.

   Ask someone to choose a number. Say that the rule is: 'If the number is even, halve it; if it is odd, add 1 and halve it.'

   Go round the class generating the chain. Write all the numbers in the chain on the board, e.g.

   Ask for a new starting number. Continue as before.

   Number chains can generate interest and challenge as it is usually not possible to guess what will happen. As more and more starting numbers are chosen, the chains can build up to a complex pattern.
   e.g. a starting number 8 joins the chain above at 4; the starting number 13 joins the chain at 7.
   a starting number of 23, goes to 12, then 6, then 3, then joins the chain at 2.

4. When finding 20% of an amount, £5.40, discuss how it is easier to first find 10% and then double it. 10% of £5.40 is 54p, so 20% of £5.40 is £1.08.

   Ask learners how they would find 5% of £5.40. Now work out 15% of £5.40.

   Encourage learners to use a range of methods for working out other percentages. They might find 15% of £15.40 by finding 10% then halving that to find 5% and adding the two together. ***Or****, having found 5% they might multiply that result by 3. They could work out 17.5% by finding 10%, 5% and 2.5% and adding all three together.*

Multiplying and dividing by multiples of 10:

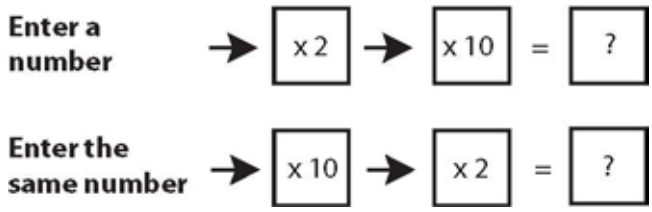
Being able to multiply by 10 and multiples of 10 depends on an understanding of place value and knowledge of multiplication and division facts. This ability is fundamental to being able to multiply and divide larger numbers.

Exemplification

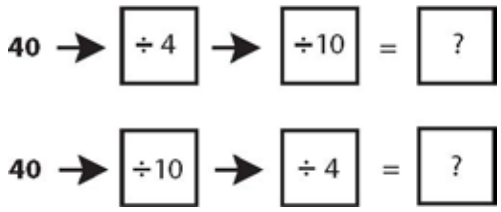
Progression	Expectations of learners with examples
	Recall multiplication and division fact for the 10 times table e.g. 7 x 10, 60 ÷ 10
	Multiply one-digit and two-digit numbers by 10 or 100 e.g. 7 x 100, 46 x 10, 54 x 100
	Change pounds to pence e.g. £1.50 to 150 pence, £6 to 600 pence
	Multiply numbers to 1000 by 10 and 100 e.g. 325 x 10, 42 x 100
	Divide numbers to 1000 by 10 and 100 (whole number answers) e.g. 120 ÷ 10, 600 ÷ 100, 850 ÷ 10
	Multiply a multiple of 10 to 100 by a single-digit number e.g. 60 x 3, 50 x 7
	Change hours to minutes; convert between units involving multiples of 10 and 100 e.g. centimetres and millimetres, centilitres and millilitres and convert between pounds and pence, metres and centimetres.
	Multiply and divide whole numbers and decimal fractions by 10, 100 or 1000 e.g. 4.3 x 10, 0.75 x 100, 25 ÷ 10, 673 ÷ 100
	Divide a multiple of 10 by a single-digit number (whole number answers) e.g. 80 ÷ 4, 270 ÷ 3
	Multiply pairs of multiples of 10 and a multiple of a 100 by a single-digit number e.g. 60 x 30, 900 x 8
	Convert larger to smaller units of measurement using decimal fractions to one place e.g. change 2.6kg to 2600g, 3.5cm to 35mm, 1.2m to 120cm
	Multiply by 25 or 50 using equivalent calculations e.g. 48 x 100 ÷ 4, 32 x 100 ÷ 2
	Multiply pairs of multiples of 10 and 100 e.g. 50 x 30, 600 x 20
	Divide multiples of 100 by a multiple of 10 and 100 e.g. 300 ÷ 50, 600 ÷ 20
	Divide by 25 or 50
	Convert between units of measurement using decimal fractions to two places e.g. change 2.75l to 2750ml or vice versa

Activities:

1. Use function machines that multiply by 10.



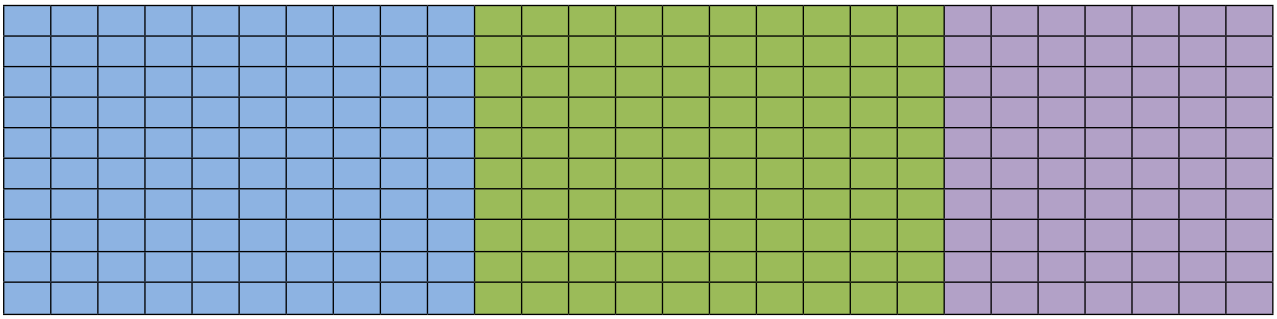
What do you notice? Try some divisions.



Try other starting numbers such as 60, 20, 80…

The function 'machine' is a useful way to focus on particular operations, in this case multiplication and then division. In the first part, learners will notice that the order of multiplication does not matter (the effect of multiplying by 10 and then by 2 is the same as multiplying by 2 and then by 10). The 'machines' can also work backwards, again exemplifying that multiplication and division are inverse operations.

2. Use a rectangular array, e.g. to show 27 x 10.



The area model is useful, especially to show how a number is partitioned into tens and units. It provides an image for learners to visualise to aid mental calculation.

3. Use a multiplication chart:

1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90
100	200	300	400	500	600	700	800	900
1000	2000	3000	4000	5000	6000	7000	8000	9000

Explain that the numbers on each row are found by multiplying the number above them by 10. So:

8 x 10 is 80, 40 x 10 is 400, and 500 x 10 is 5000.

If you skip a row, the numbers are multiplied by 100, so:

2 x 100 is 200, 70 x 100 is 7000.

Use the chart for dividing:

50 ÷ 10 = 5, 600 ÷ 10 = 60, and 4000 ÷ 100 = 40.

Extend the chart to show decimals by inserting decimals 0.1 to 0.9 above the numbers 1 to 9.

This chart is very helpful for showing multiplication by power of 10. Going down a row has the effect of multiplying by 10, while going down two rows produces a multiplication by 100. Similarly, it demonstrates that multiplication and division are inverse operations. Going up a row represents division by 10 and two rows division by 100.

4. Use a multiplication grid. Ask learners to find the missing numbers.

x	2		7
	40		
10		50	

Learners are probably familiar with a conventional tables square. It can be more interesting to be given smaller grids in a random order.

Here learners must recognise that the first vertical column is part of the 2 times table; the number in the cell below the 40 must be found by 2 x 10 = 20. They can deduce that the middle column must be times 5 in order to get the 50.

Multiplying and dividing by single-digit numbers and multiplying by two-digit numbers:

Once learners are familiar with some multiplication facts, they can extend their skills.

- One strategy is to partition one of the numbers and use the distributive law of multiplication over addition.

  So, for example, e.g. 12 x 7 = (10 + 2) x 7 = 10 x 7 + 2 x 7 = 84.

  Subtraction can be used similarly, so 'nine eights are ten eights minus one eight'.
- Another strategy is to make use of factors, so 7 x 6 is seen as 7 x 3 x 2.

Once learners understand the effect of multiplying and dividing by 10, they can start to extend their multiplication and division skills to larger numbers.

- A product such as 26 x 3 can be worked out by partitioning 26 into 20 + 6, multiplying each part by 3, then recombining.
- One strategy for multiplication by 2, 4, 8, 16, 32, … is to use doubling, so that 9 x 8 is seen as 9 x 2 x 2 x 2. A strategy for dividing by the same numbers is to use halving.
- A strategy for multiplying by 50 is to multiply by 100, then halve, and for multiplying by 25 is to multiply by 100 then divide by 4.

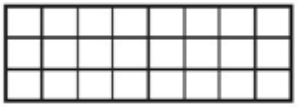
Since each of these strategies involves at least two steps, most learners will find it helpful to ***make jottings*** of the intermediate steps in their calculations.

Exemplification

Progression	Expectations of learners with examples
	Find one quarter by halving one half
	Multiply numbers to 20 by a single-digit number e.g. 17 x 3
	Multiply and divide two-digit numbers by 4 or 8 e.g. 26 x 4, 96 ÷ 8
	Multiply two-digit numbers by 5 or 20 e.g. 32 x 5, 14 x 20
	Multiply by 25 or 50 e.g. 48 x 25, 32 x 50
	Multiply a two-digit and a single digit number e.g. 28 x 7
	Divide a two-digit number by a single-digit number e.g. 68 ÷ 4
	Divide by 25 or 50 e.g. 480 ÷ 25, 3200 ÷ 50
	Find new facts from given facts e.g. given that three pears cost 24p, find the cost of 4 pears.

Activities:

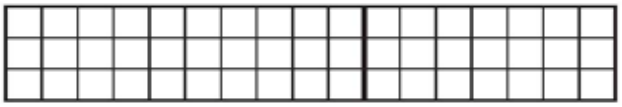
1. Use an area model for simple multiplication facts (e.g. show 8 x 3 as):



How many … rows? columns? small squares?

Encourage children to visualise other products in a similar way.

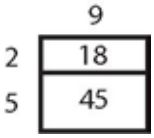
Extend this model to larger numbers, such as 17 x 3: split the 17 into 10 + 7 and use 10 x 3 + 7 x 3.



How many … rows? columns? small squares?

The rectangles give a good visual model for multiplication: the areas can be found by repeated addition (in the case of the first example, 8 + 8 + 8), but learners should then commit 3 x 8 to memory and know that it gives the same answer as 8 x 3.

2. Use multiplication facts that children know in order to work out others. For example, knowing 9 x 2 and 9 x 5, work out 9 x 7.



Area models like this discourage the use of repeated addition. The focus is on the separate multiplication facts. The diagram acts as a reminder of the known facts, which can be entered in the rectangles, and the way that they are added in order to find the answer.

3. Discuss the special cases of multiplying by 25 and 50, which are easily done by multiplying by 100 and dividing by 4 or 2 respectively.

The use of factors often makes a multiplication easier to carry out.

4. Use factors. For example, work out 13 x 12 by factorising 12 as 3 x 4 or 6 x 2:

13 x 12	13 x 12	13 x 12	13 x 12
3 4	3 2 2	2 6	2 2 3
13 x 3 x 4	13 x 3 x 2 x 2	13 x 2 x 6	13 x 2 x 2 x 3

Discuss which are easiest to use.

Diagrams like this can help learners to keep track of the separate products.

Fractions, decimal fractions and percentages:

Learners need an understanding of how fractions, decimal fractions and percentages relate to each other. For example, if they know that ½, 0.5 and 50% are all ways of representing the same part of the whole, then they can see that the calculations:

half of 40, ½ x 40, 40 x ½, 40 x 0.5, 0.5 x 40, 50% of 40

are different versions of the same calculation. Sometimes it might be easier to work with fractions, sometimes with decimal fractions and sometimes with percentages.

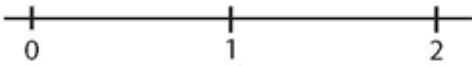
There are strong links between this section and the earlier section 'Multiplying and dividing by multiples of 10'.

Exemplification

Progression	Expectations of learners with examples
	Find half of any even number to 40 or multiple of 10 to 100 e.g. halve 80
	Find half of any multiple of 10 up to 200 e.g. halve 170
	Find $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ and $\frac{1}{10}$ of numbers in the 2, 3, 4, 5 and 10 times tables
	Find half of any even number to 200
	Find unit fractions and simple non-unit fractions of whole numbers or quantities e.g. $\frac{3}{8}$ of 24
	Recall fractions and decimal fraction equivalents from one-half, quarters, tenths and hundredths e.g. recall the equivalence of 0.3 and $\frac{3}{10}$ and 0.03 and $\frac{3}{100}$
	Recall percentage equivalents of one-half, one-quarter, three-quarters, tenths and hundredths
	Find fractions of whole numbers or quantities e.g. $\frac{2}{3}$ of 27, $\frac{4}{5}$ of 70kg
	Find 50%, 25% or 10% of whole numbers or quantities e.g. 25% of 20kg, 10% 0f £80
	Recall equivalent fractions, decimal fractions and percentages for hundredths e.g. 35% is equivalent to 0.35 or $\frac{35}{100}$
	Find half of decimal fractions with units and tenths e.g. half of 3.2
	Find 10% or multiples of 10% of whole numbers and quantities e.g. 30% of 50ml, 40% of £30, 70% of 200g
	Recall commonly used equivalent fractions for $33\frac{1}{3}$% and $66\frac{2}{3}$%

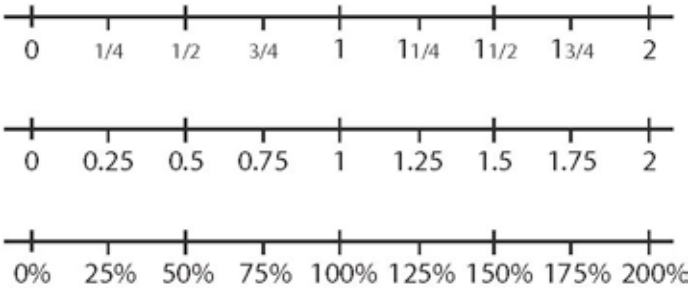
Activities:

1. Draw a number line marking on it the points 0, 1 and 2



Ask learners to show where the fractions ¼, ½, 1¼, 1½, 1¾ fit on the line. Ask what other fractions between 0 and 2 they could add to the line.

When they are familiar with fractions, draw a new line under the first one and ask for the decimal fractions 0.5, 1.5, 0.25, 1.25, 1.75, 0.75 to be placed on this line. Repeat with a line for percentages from 0% to 200%



Discuss the equivalence of, for example, ¼, 0.25 and 25%. Choose any number and ask children to call out the equivalents on the other two lines.

Number lines are useful for showing fractions as points on the number line between the whole numbers. This helps to move learners from the idea of fractions as parts of shapes. The first part of the activity requires them to think about the relative size of fractions. Separate number lines with, for example, halves, quarters and eighths placed on one line under the other, help to establish the idea of equivalent fractions. Gradually the lines can be built up to include more fractions, for example, twelfths or twentieths. As in the example above, they can also be used to demonstrate the equivalence between fractions, decimal fractions and percentages.

2. Write a sum of money on the board/whiteboard etc e.g. £24

Ask children in turn to say what half of £24 is, then $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$ and $\frac{1}{12}$.

Then give fractions such as $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$ and $\frac{5}{12}$. Ask how learners could calculate these fractions of £24.

To answer questions like these learners will use the basic strategy of using what they already know to work out related facts. In this case, they will need to know how to find unit fractions (with a numerator of 1) of an amount and use this to find other fractions.
For example, knowing that $\frac{1}{3}$ of £24 is £8 they can work out $\frac{2}{3}$ is twice as much, or £16. Similarly, knowing that $\frac{1}{8}$ of £32 is £4, then $\frac{3}{8}$ is three times as much.

3. Give learners a percentage example, say, 25% of £60. Discuss different ways of interpreting the question such as $\frac{25}{100}$ of £60 or $\frac{1}{4}$ of £60.
Learners calculate $\frac{1}{4}$ of £60 by saying that $\frac{1}{2}$ of £60 is £30 and $\frac{1}{2}$ of £30 is £15.
Alternatively they might calculate 10% of £60, which is £6, so 5% of £60 is £3 and 20% of £60 is £12, so 25% of £60 is £12 + £3 or £15.

Ask learners to find 17.5% of £60. Since they know that 5% of £60 is £3, they can work out that 2.5% of £60 is £1.50. They can then find the total by adding 10% + 5% + 2.5%.

"Percentage Bubble" diagrams are a useful visual tool to reinforce and highlight this strategy.

Aim to give learners examples that they have seen in everyday life, such as from newspapers or local shops, to give a more realistic and motivating context.

It is important that learners understand the basic fact that % means 'out of a hundred' or 'per hundred' rather than to learn any rules about working with percentages. Examples of percentage calculations that can be done mentally can usually be worked out using this basic knowledge.

Further Reading:

Scottish Survey of Literacy and Numeracy Report 2012 - **http://glo.li/WXaodF**

Learning Together – Mathematics (HMI) - **http://glo.li/14xKNpp**

How Are Learners Progressing and Achieving in Numeracy? - **http://glo.li/14xK6wm**

Dumfries and Galloway Numeracy Strategy (3-18) - **http://glo.li/WXcUkl**

Assessing Progress and Achievement in Numeracy and Mathematics - **http://glo.li/16Sm4BC**

Acknowledgements:

- This Mental Strategies document has been adapted from *a National Strategies Primary (Department of Education England and Wales)* publication entitled ***'Teaching Learners To Calculate Mentally (2010)'***

  The above mentioned paper has been the basis of much of our work here and was an invaluable resource in putting together the Dumfries and Galloway Mental Maths Strategies for schools.

- New Zealand Number Framework

The hard work of the Dumfries and Galloway Numeracy Working Group was also instrumental in developing and compiling this guidance for schools.

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